Combinatorial Perpetual Scheduling

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joint work with Mirabel Mendoza, Arturo Merino, and Mads Anker Nielsen

schedulingseminar.com
March 18, 2026

Bamboo Garden Trimming

or: Perpetual Maintenance Scheduling

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[Gasieniec et al., SOFSEM'17]

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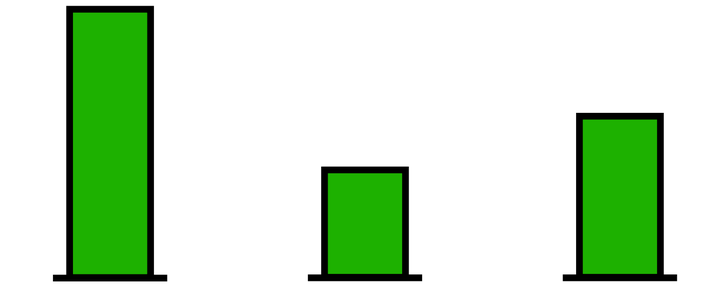

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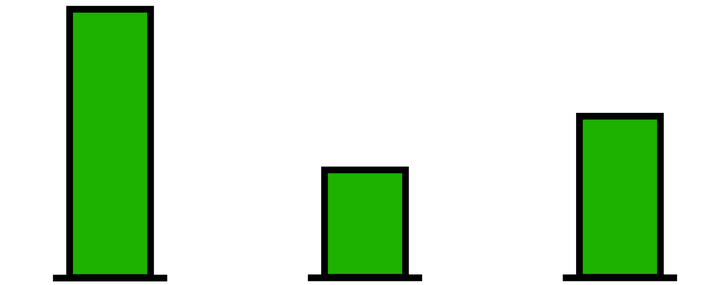

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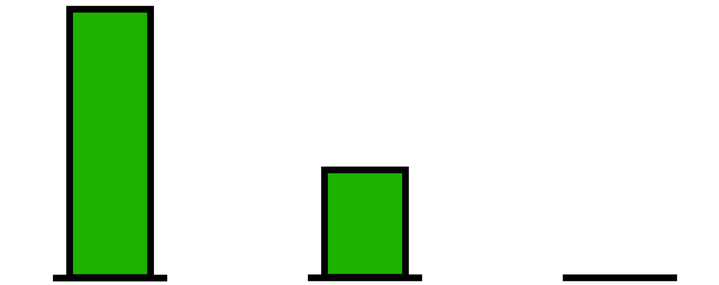

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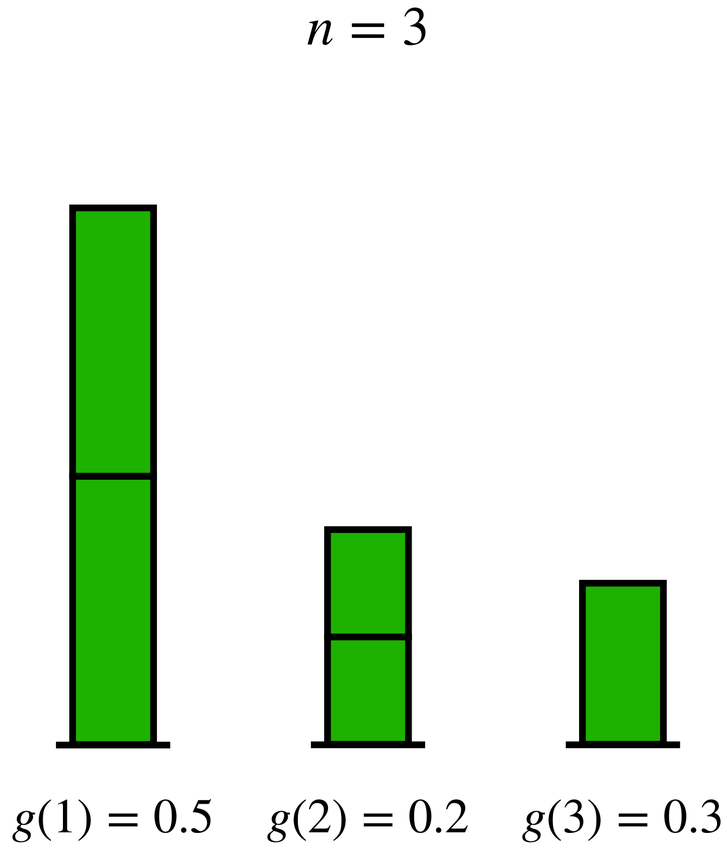

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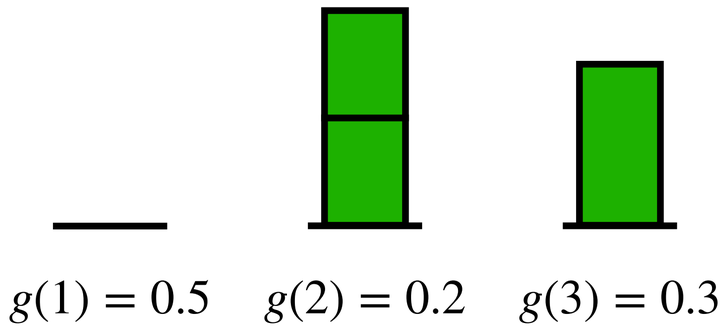

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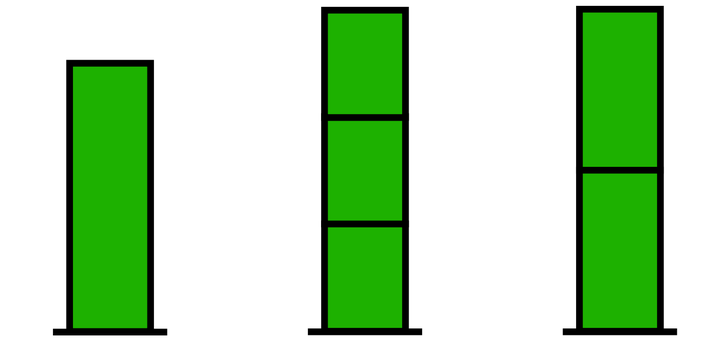

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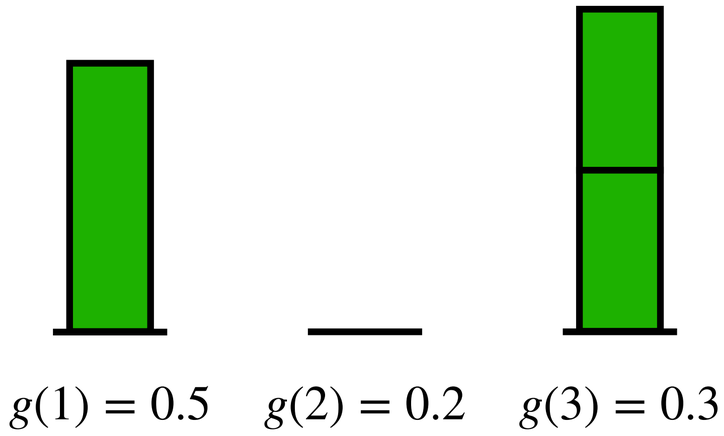

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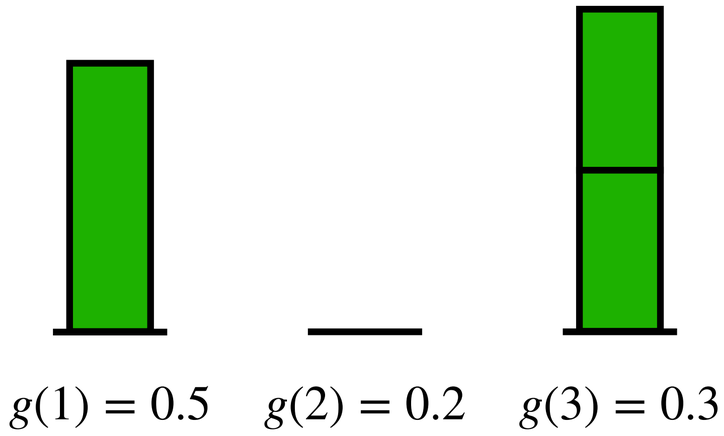

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- Normalization: assume $\sum_{e=1}^{n} g(e) = 1$.

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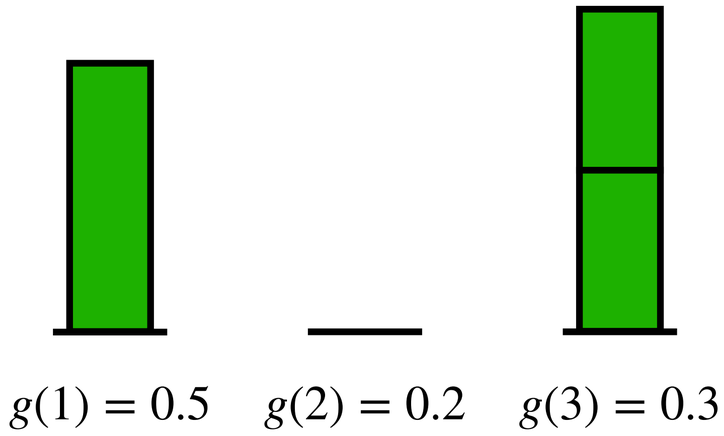

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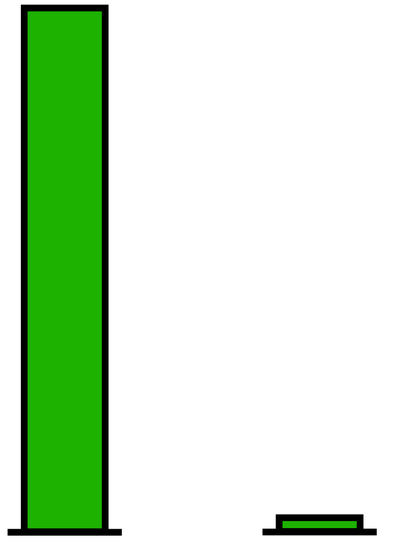

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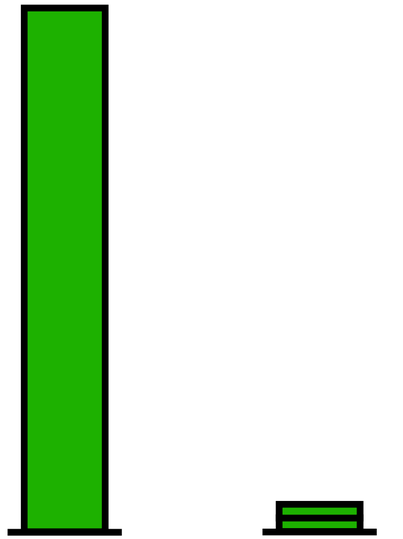

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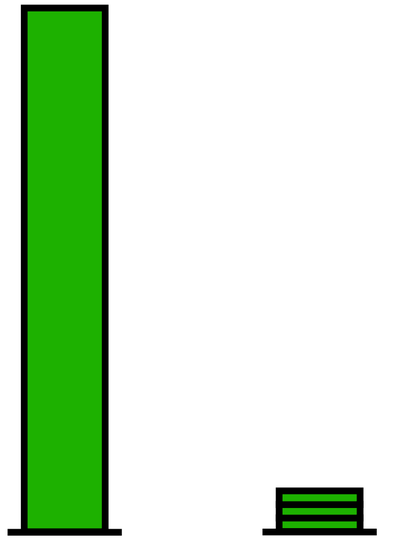

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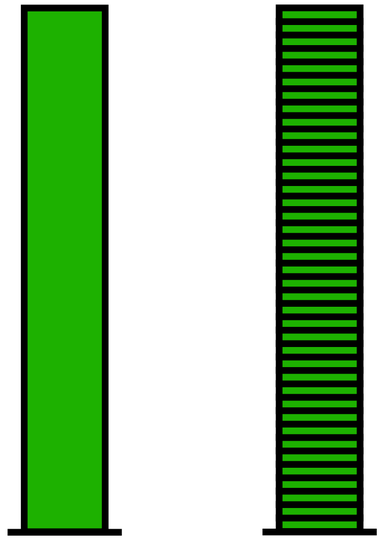

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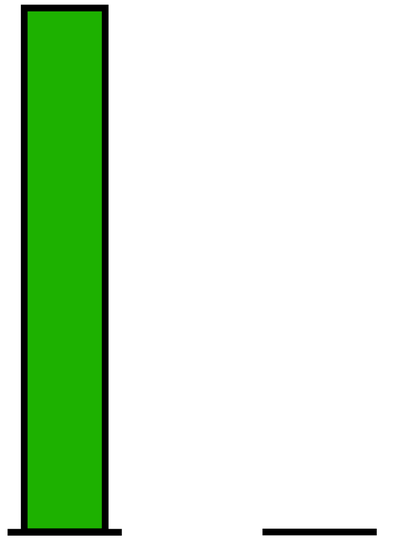

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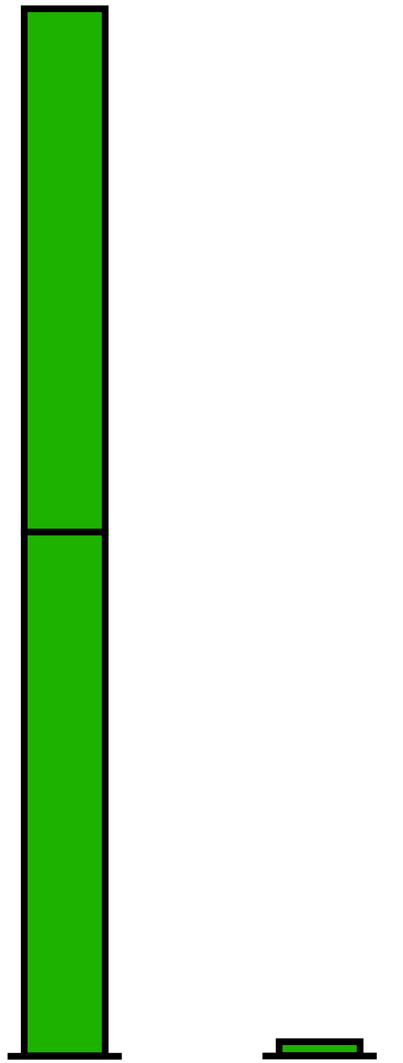

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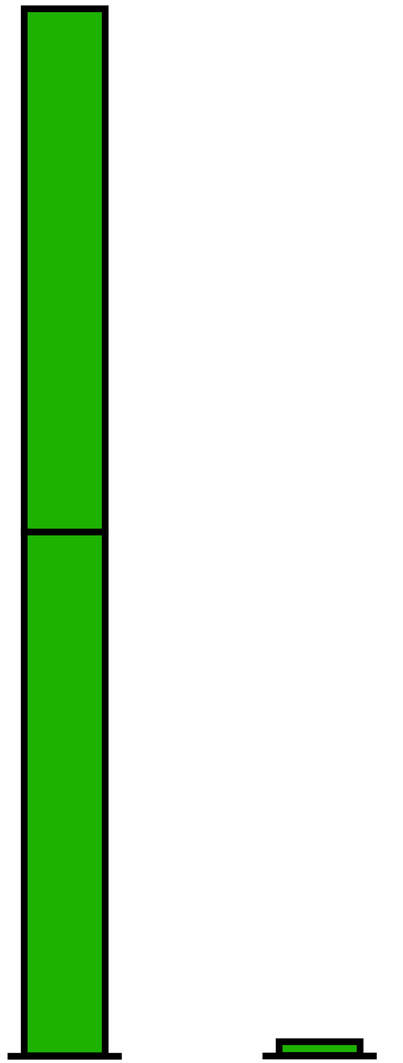

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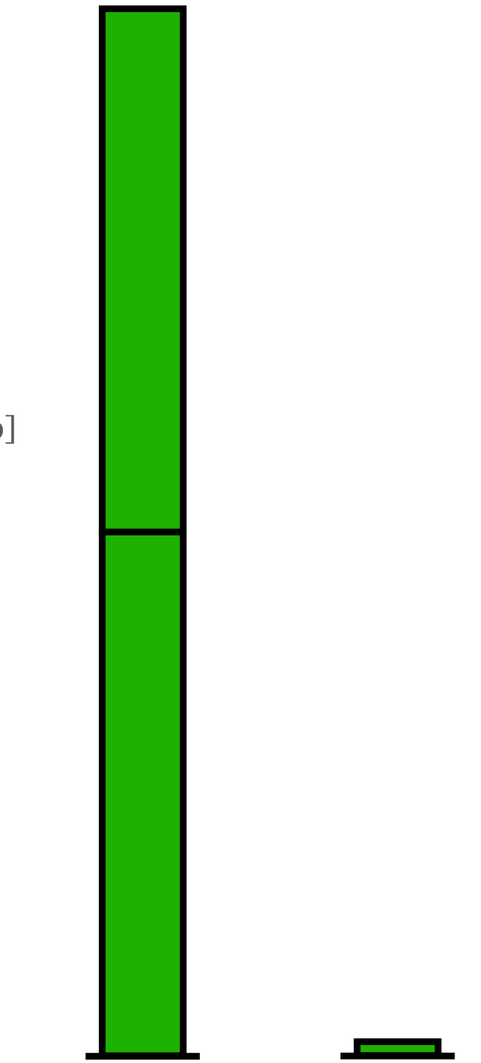

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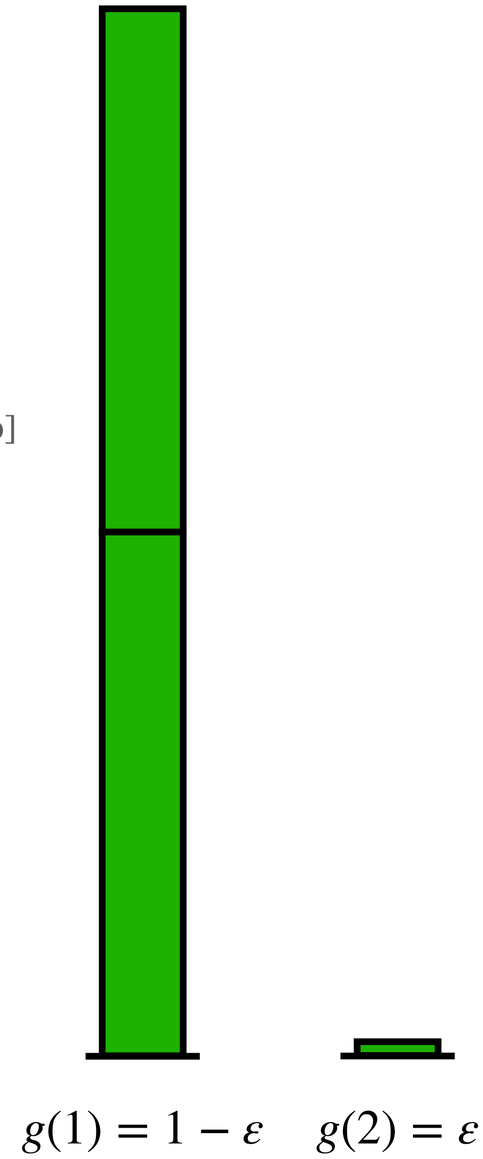

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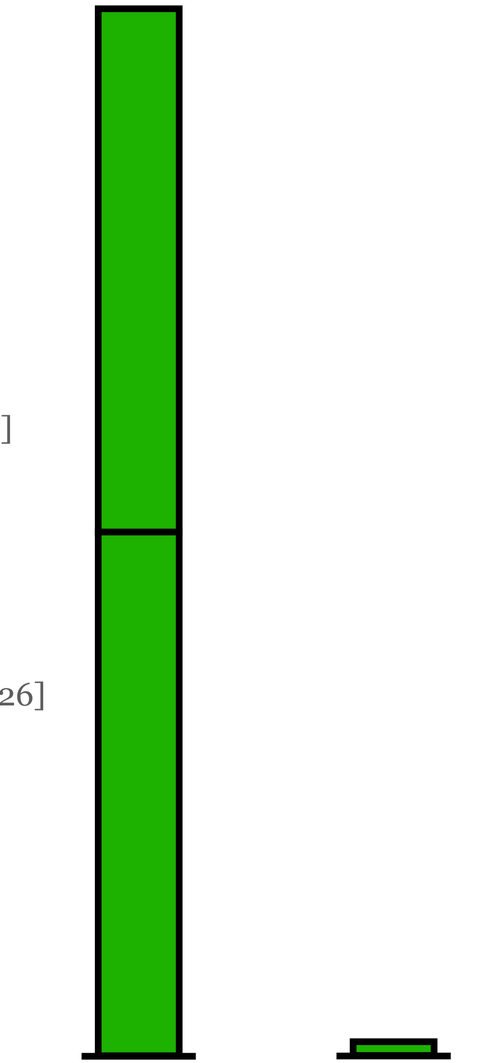

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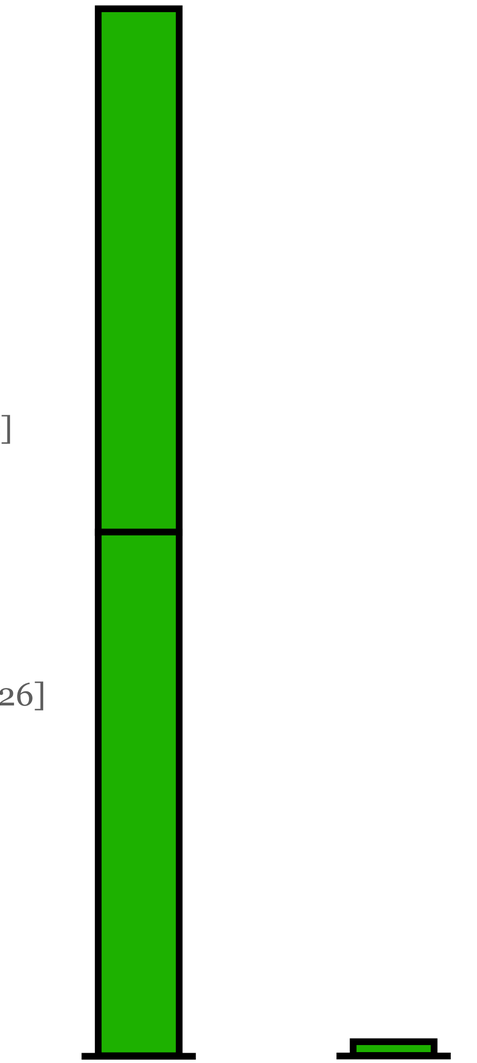

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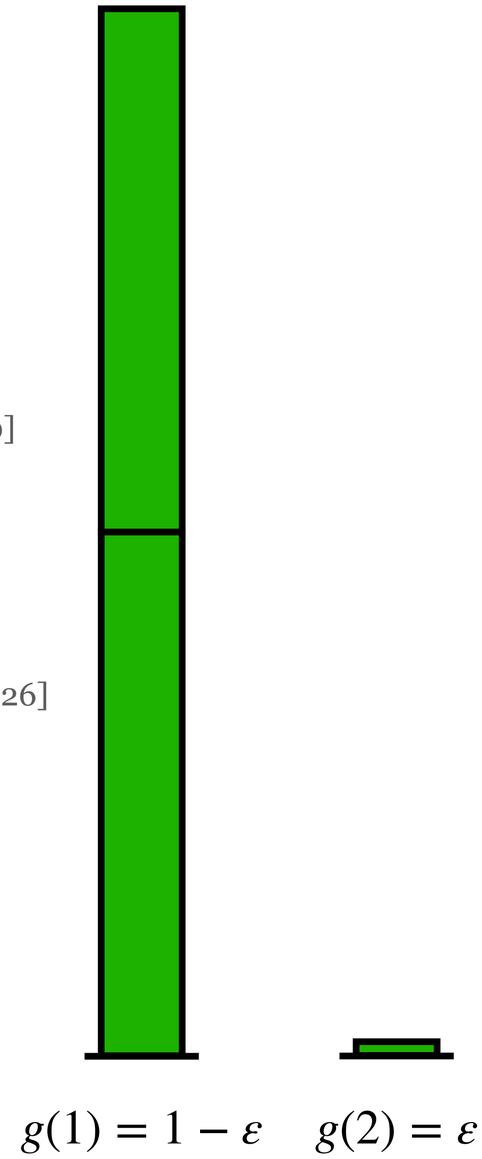

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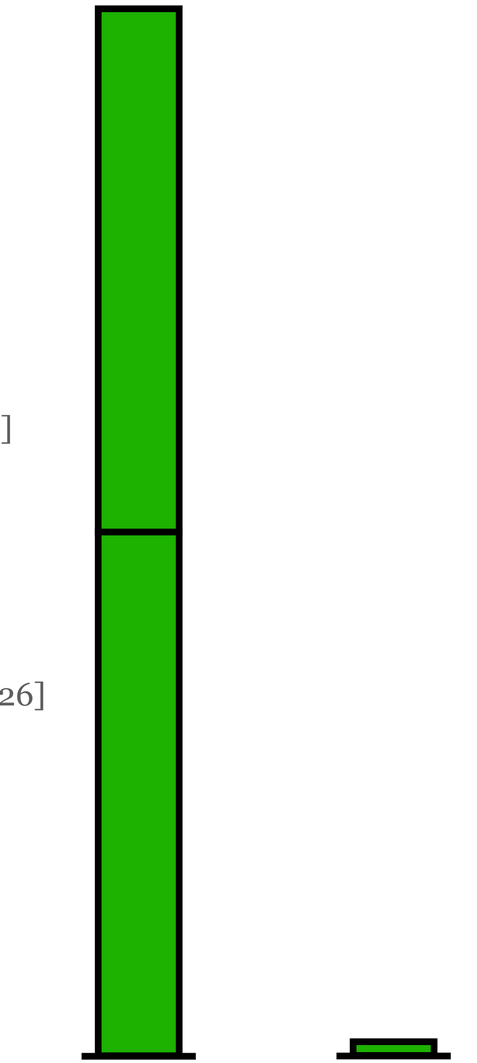

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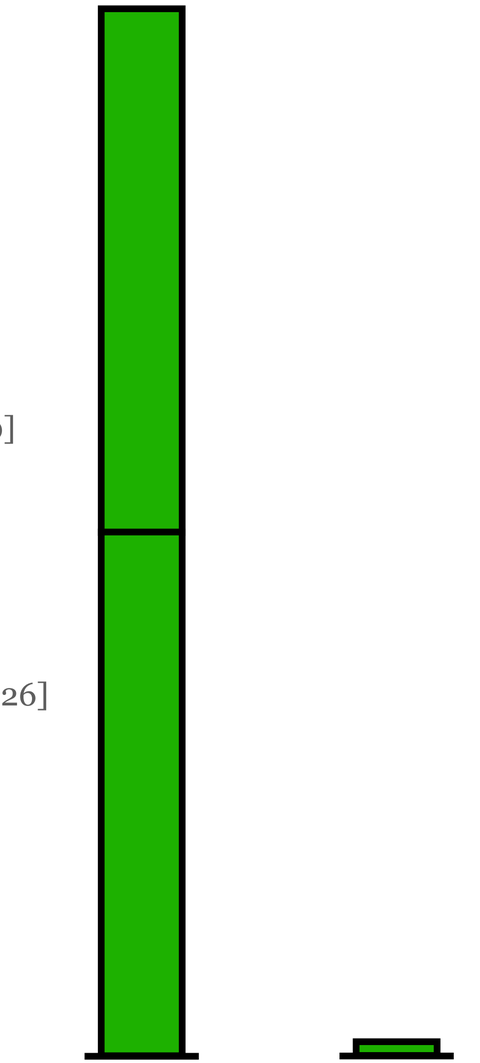

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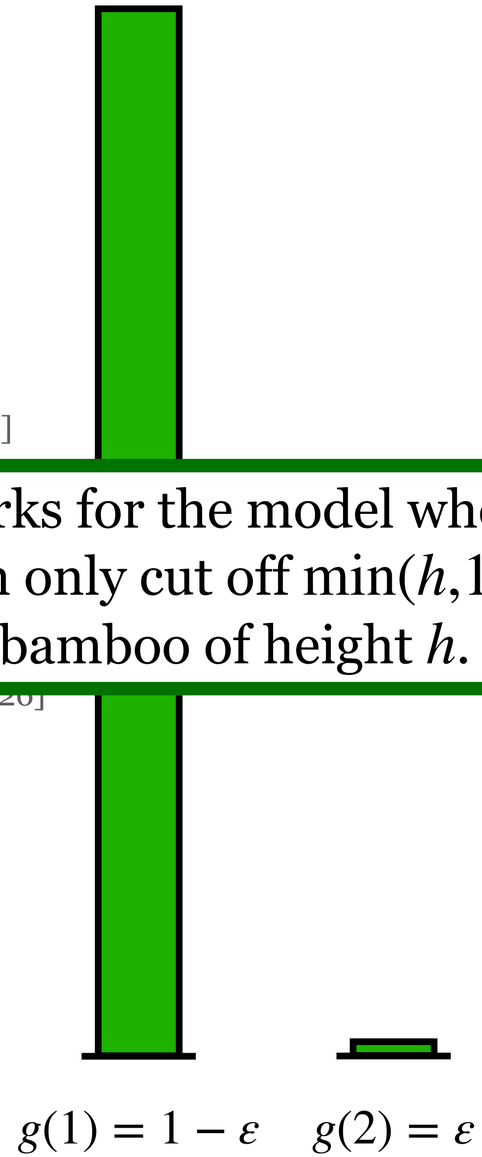

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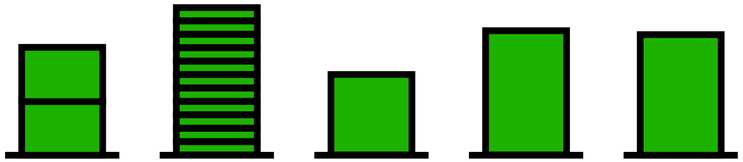

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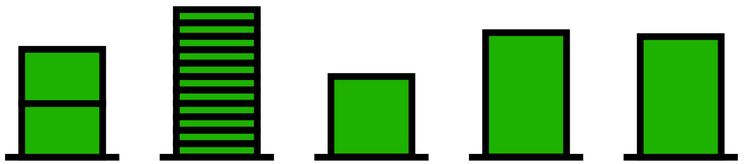

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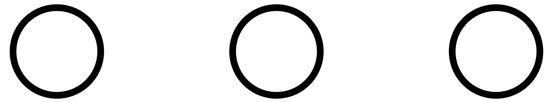

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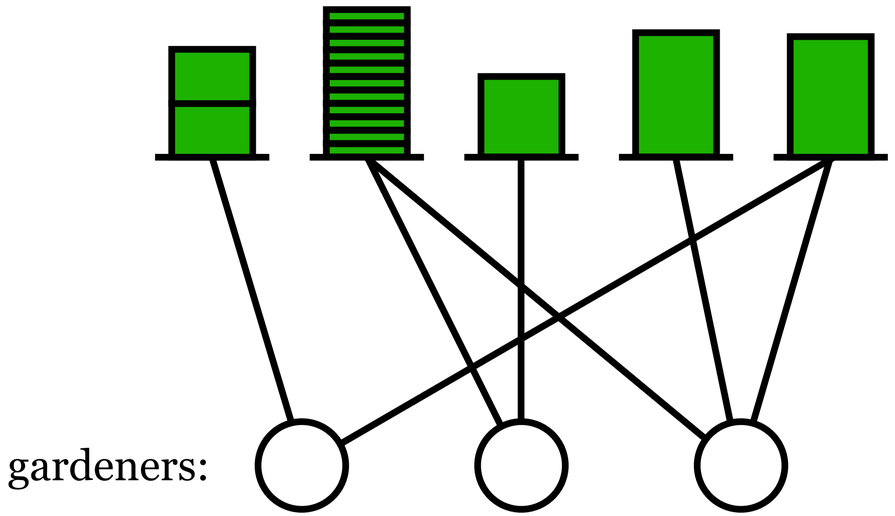

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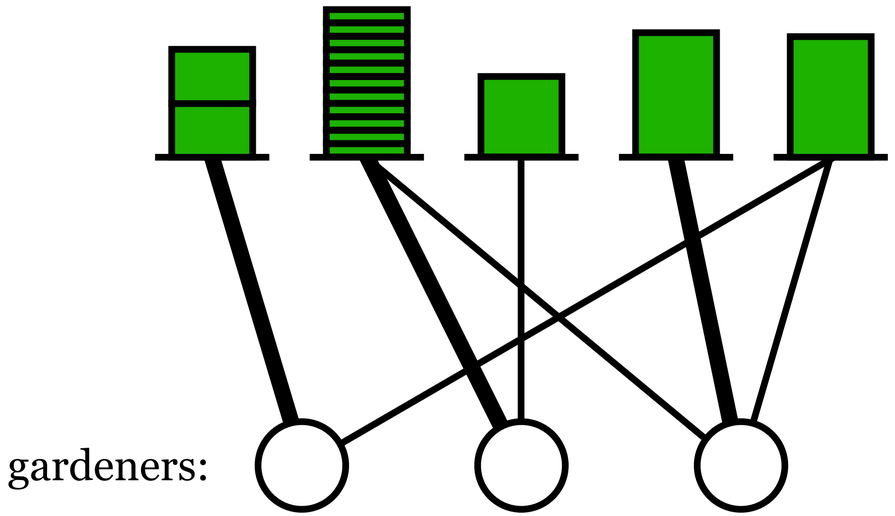

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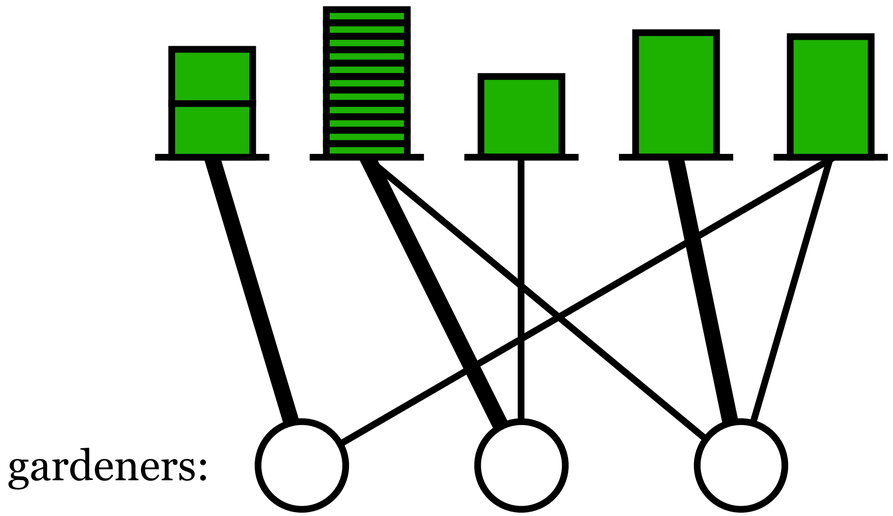

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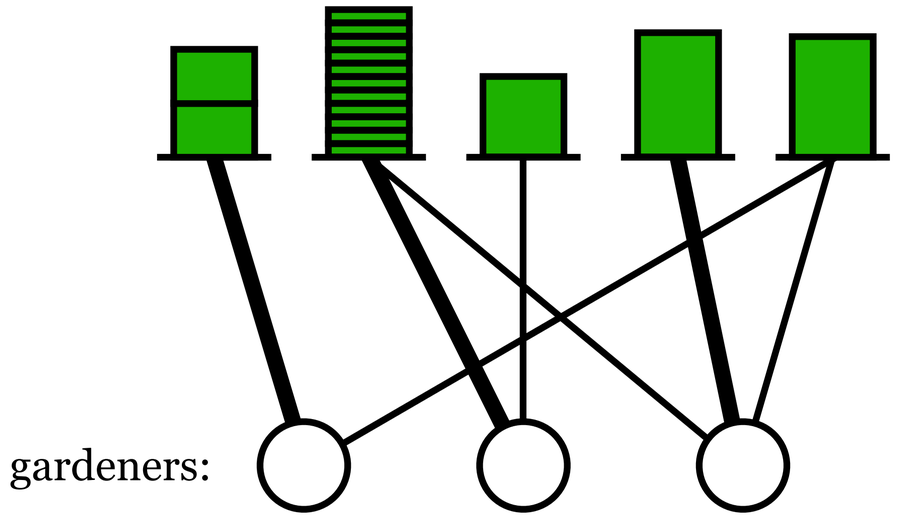

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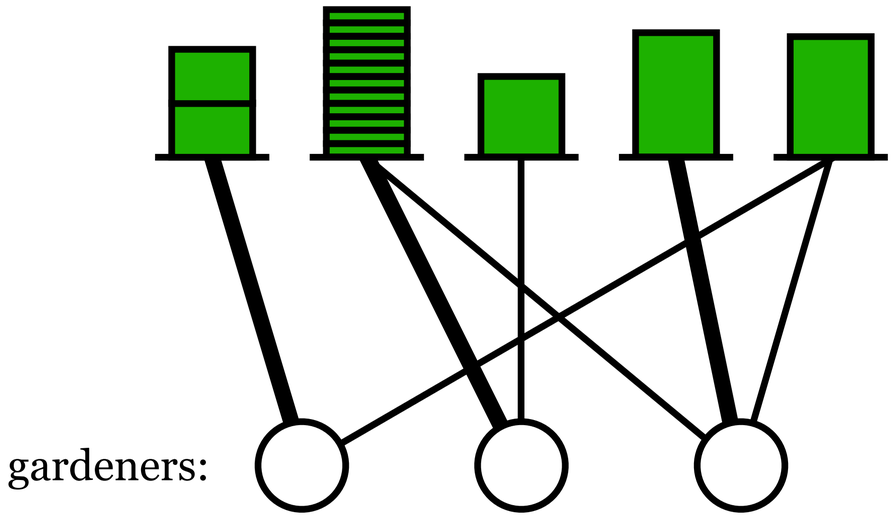

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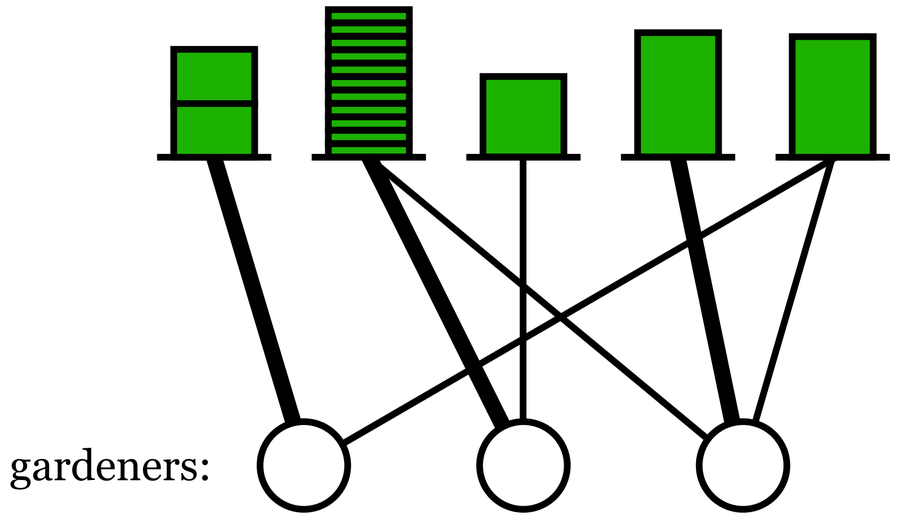

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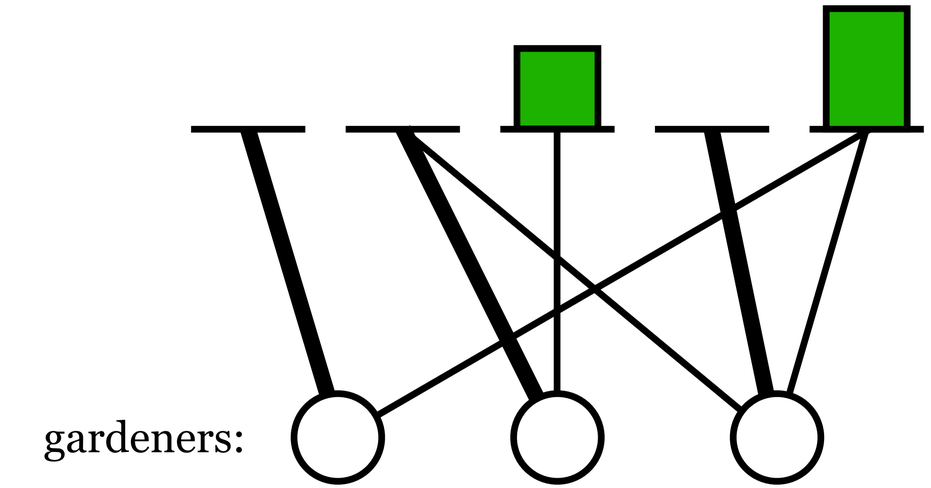

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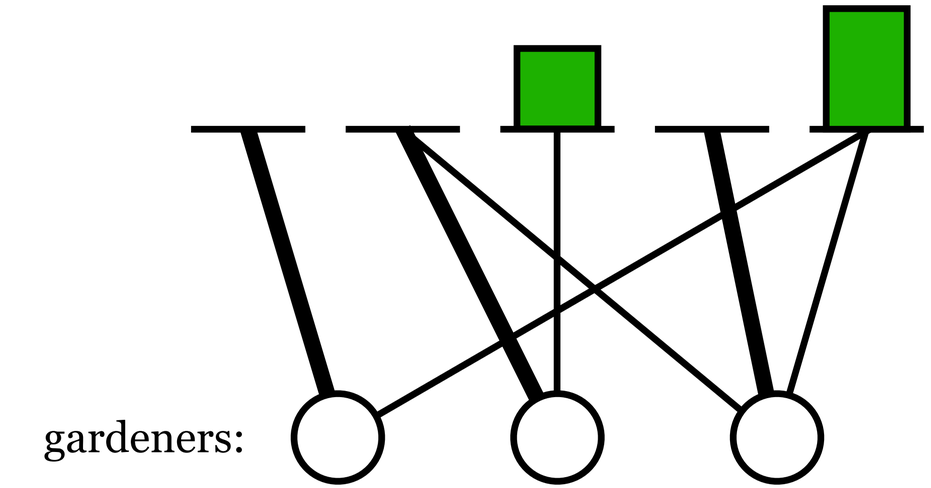

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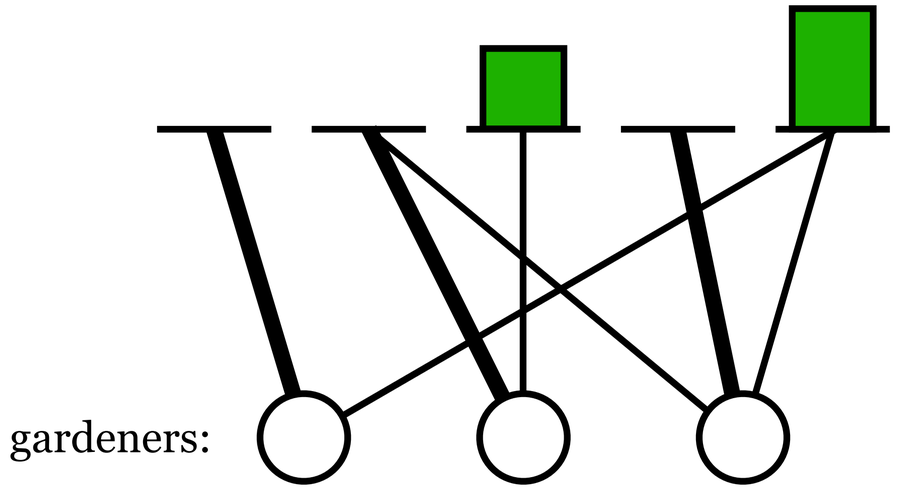

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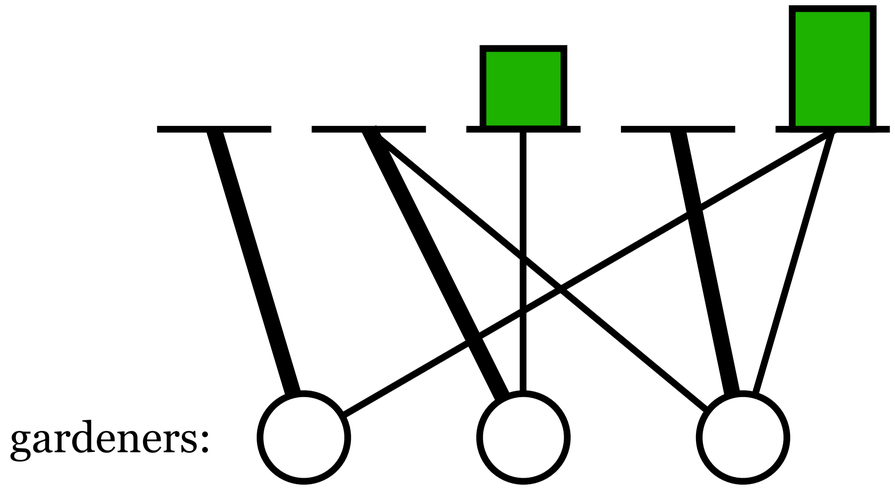

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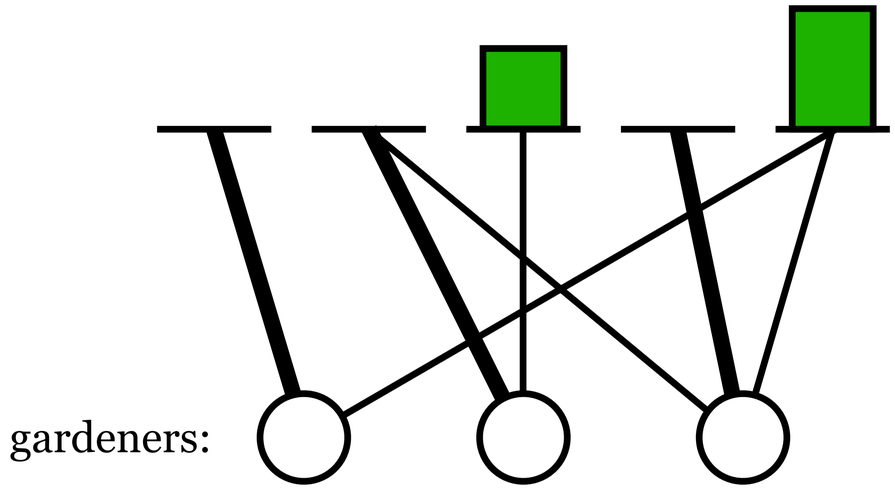

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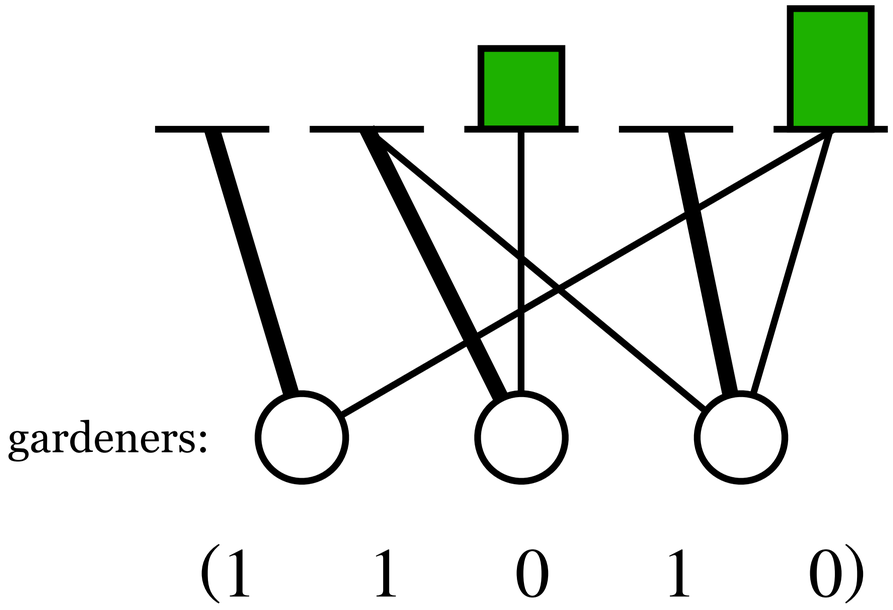

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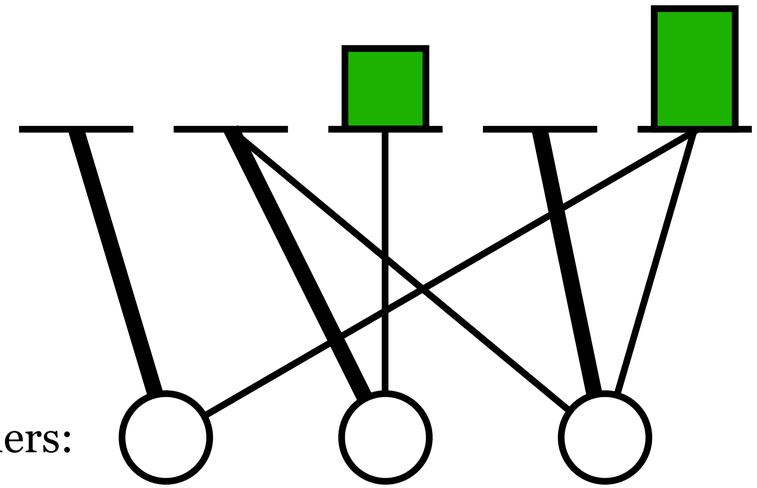

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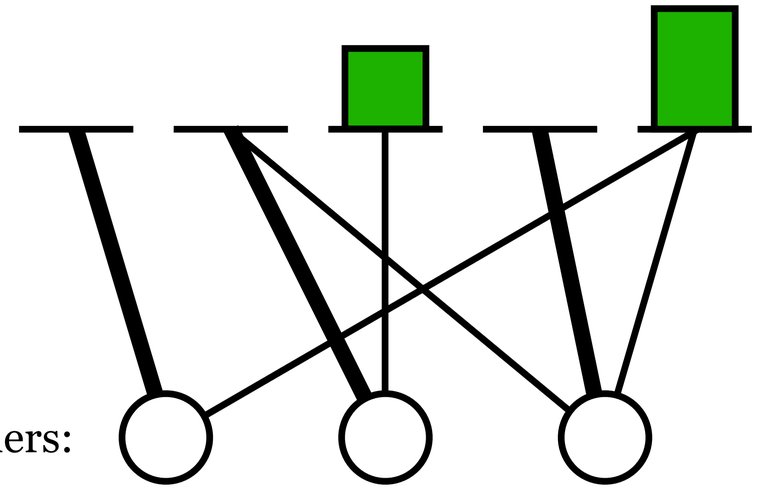

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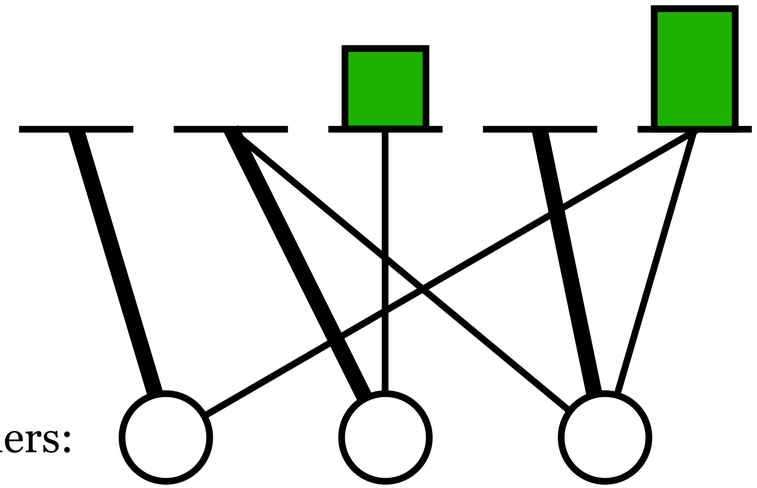

$$
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\hline
g: & (0.4 & 0.1 & 0.6 & 1 & 0.9)
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- Similar extensions have, e.g., also been considered for variants such as the adversarial variant (cup games).

A Lower Bound

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$$n = \binom{2k}{k} \text{ bamboos}$$

$$\begin{pmatrix} 0 & 1 & & 1 & 1 \\ 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & & 1 & 0 \\ 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & & 0 & 1 \end{pmatrix} \quad 2k \text{ allowed sets}$$

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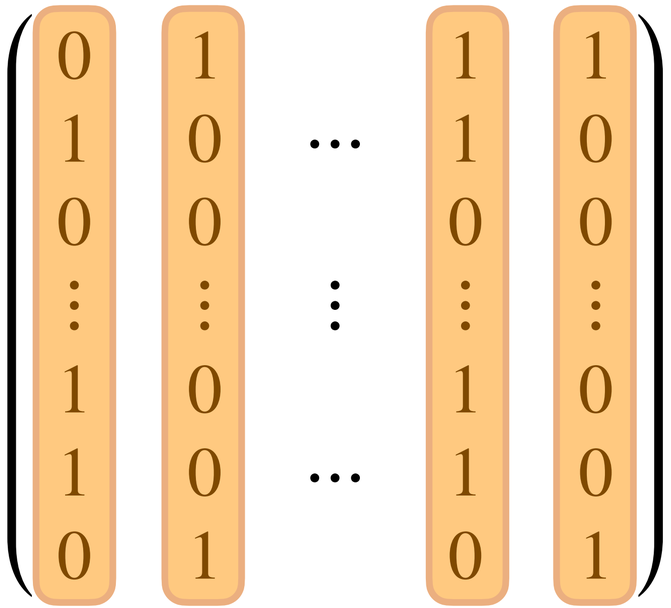

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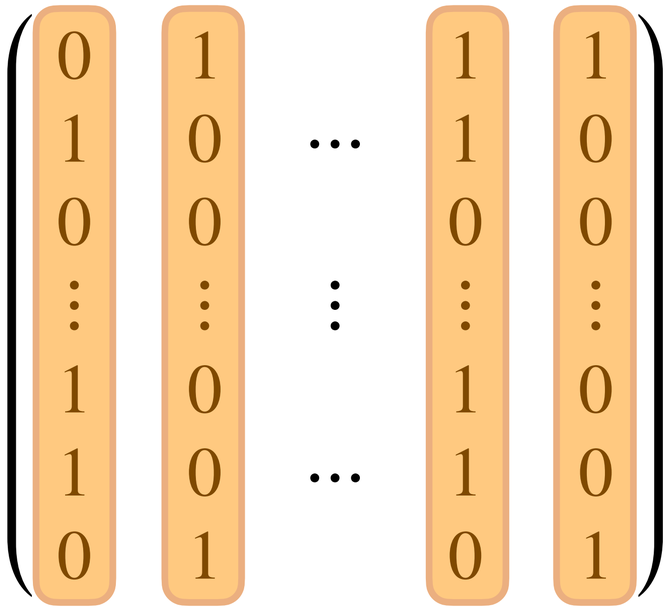

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Conclusion: Can guarantee precisely $\Theta(\log n)$.

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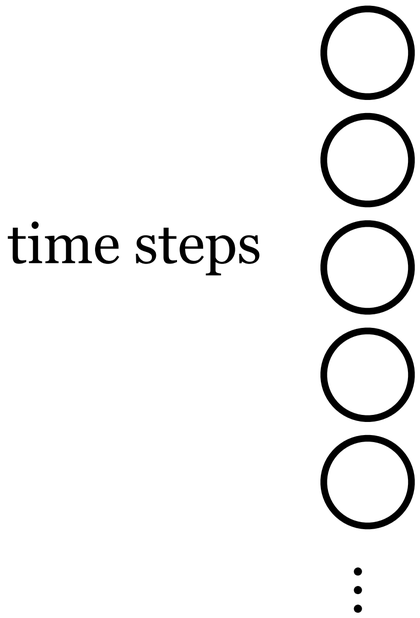

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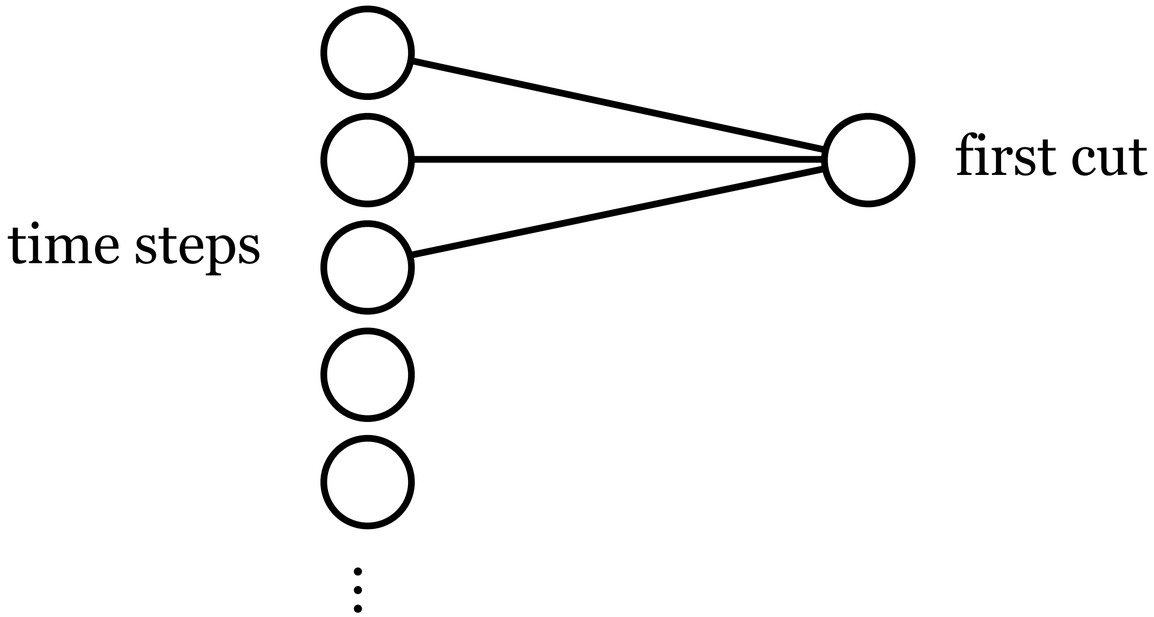

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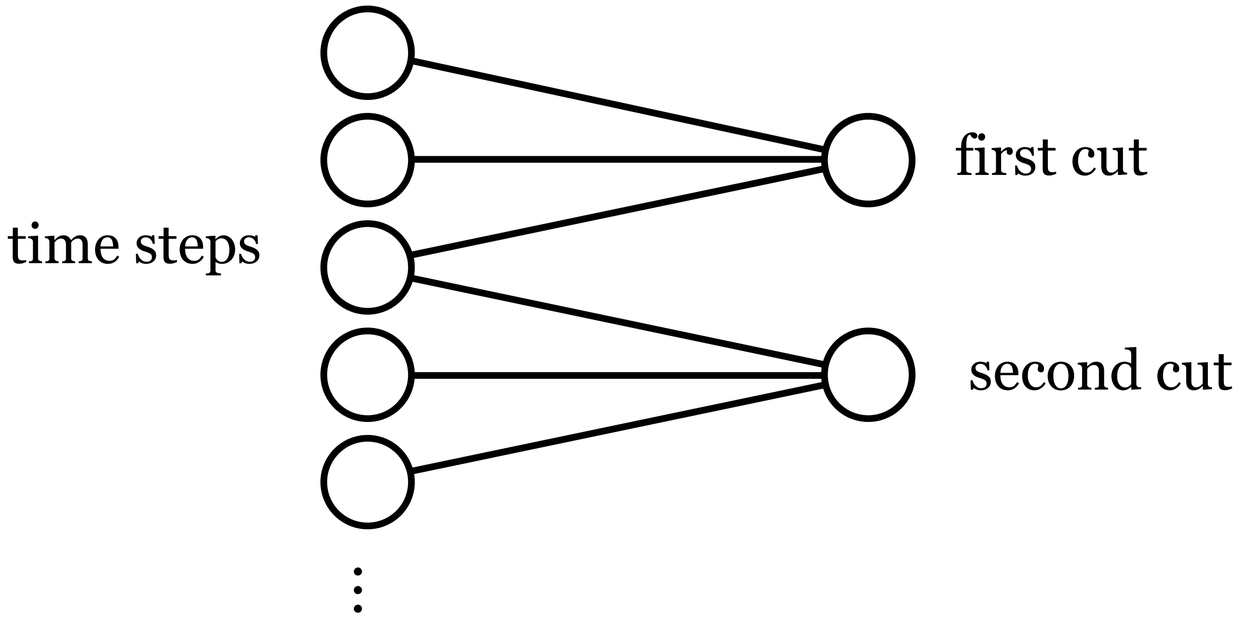

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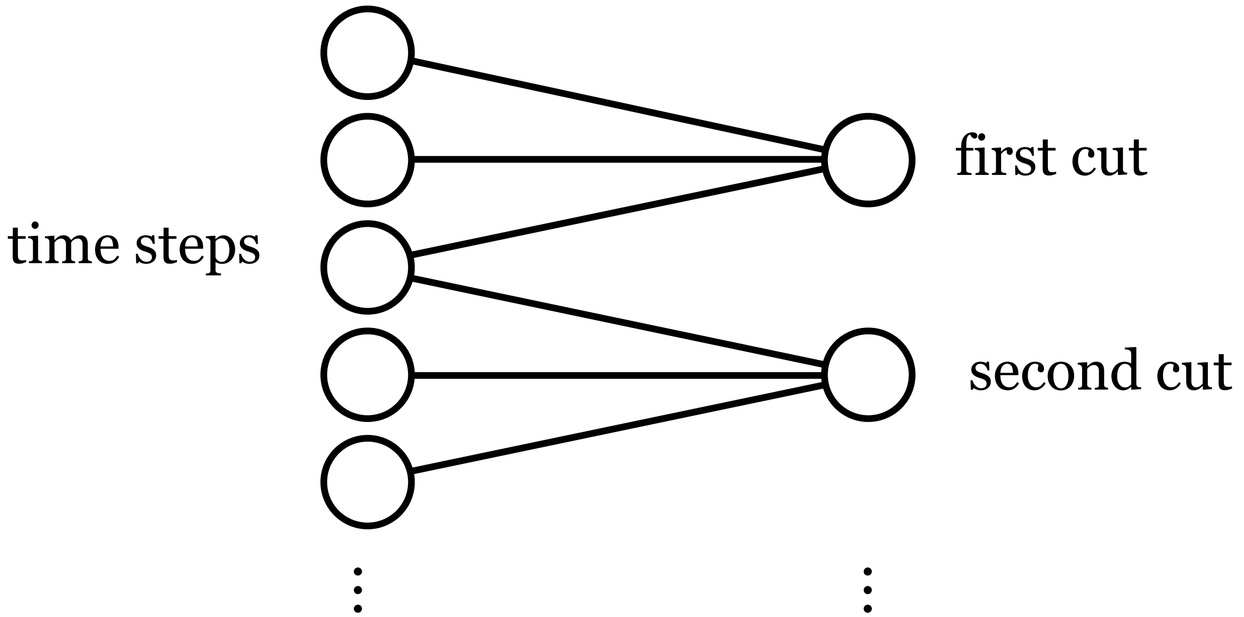

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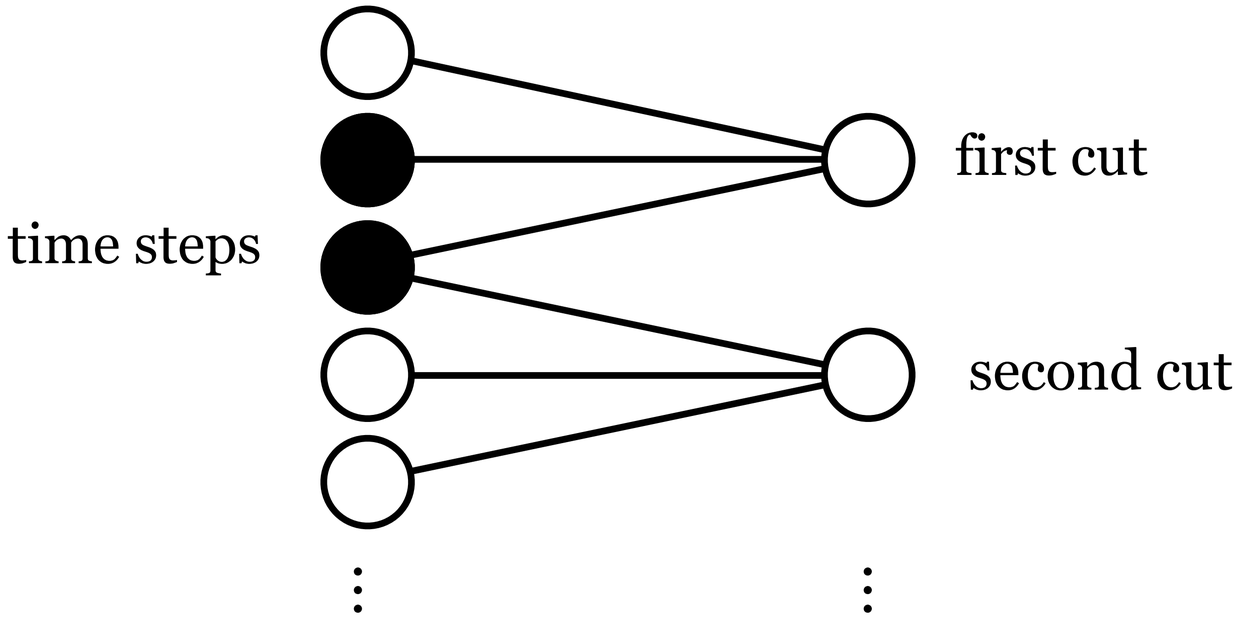

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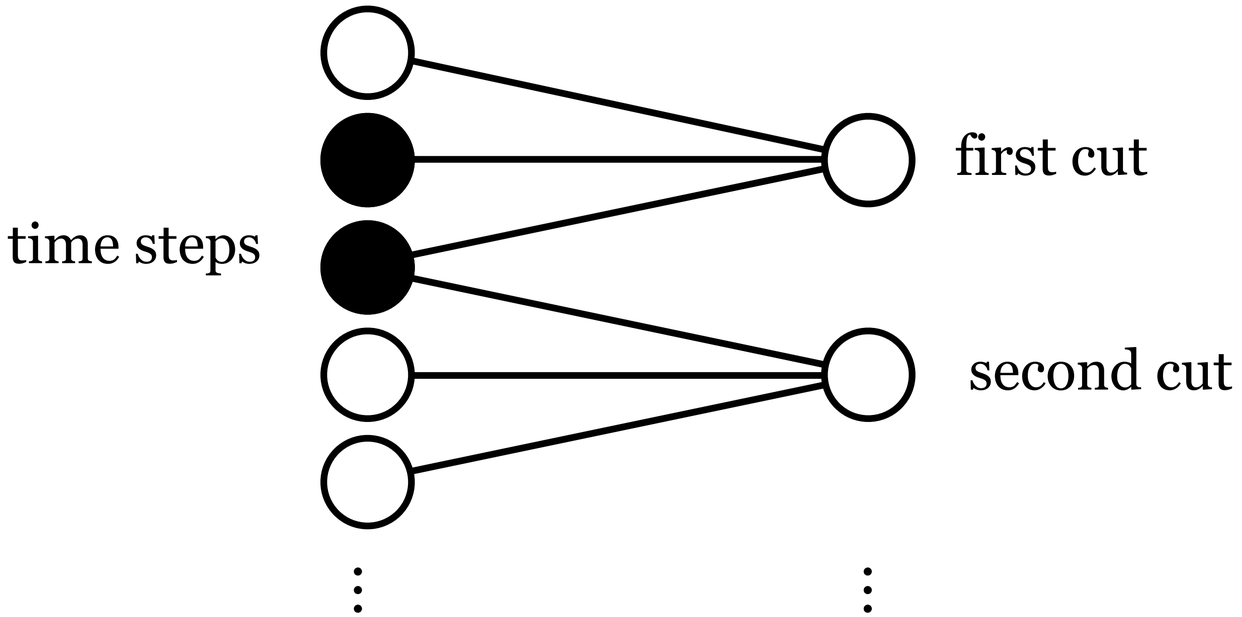

Thus: Feasible (in above sense) iff we cut according to basis of above transversal matroid M_e.

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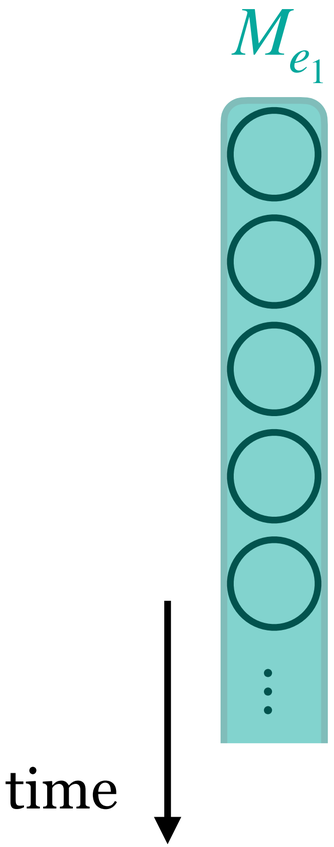

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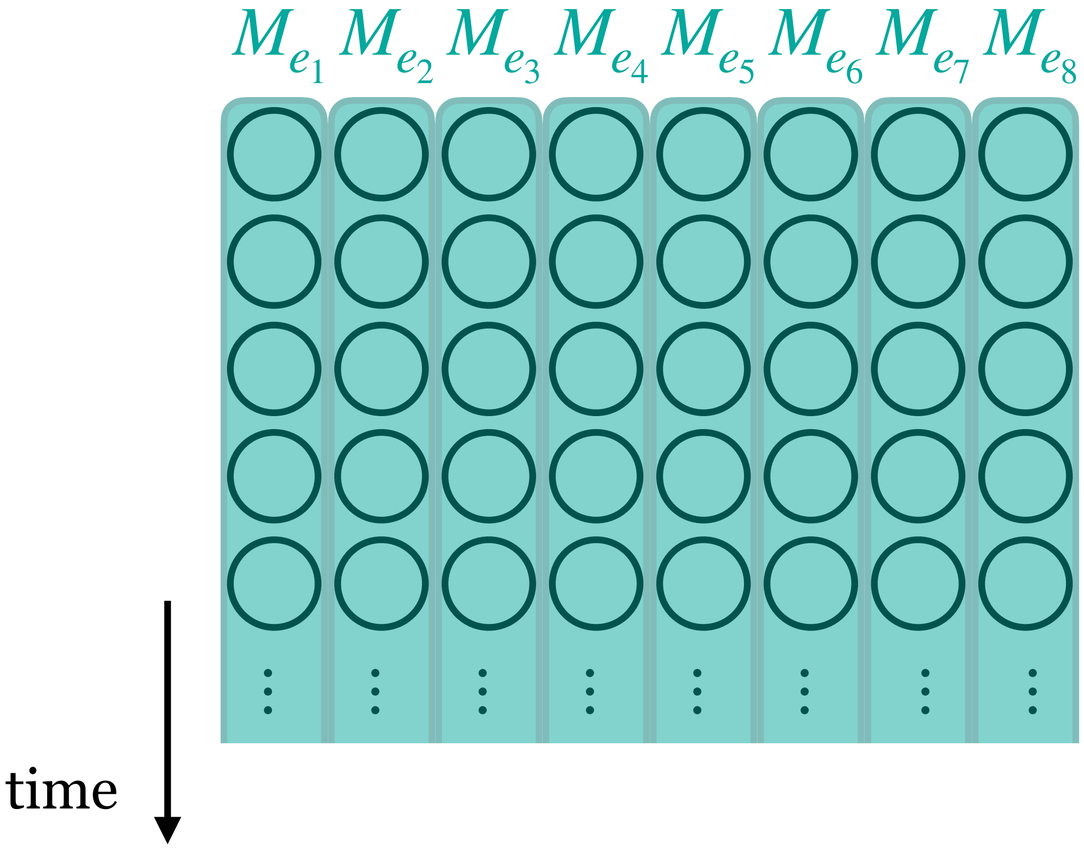

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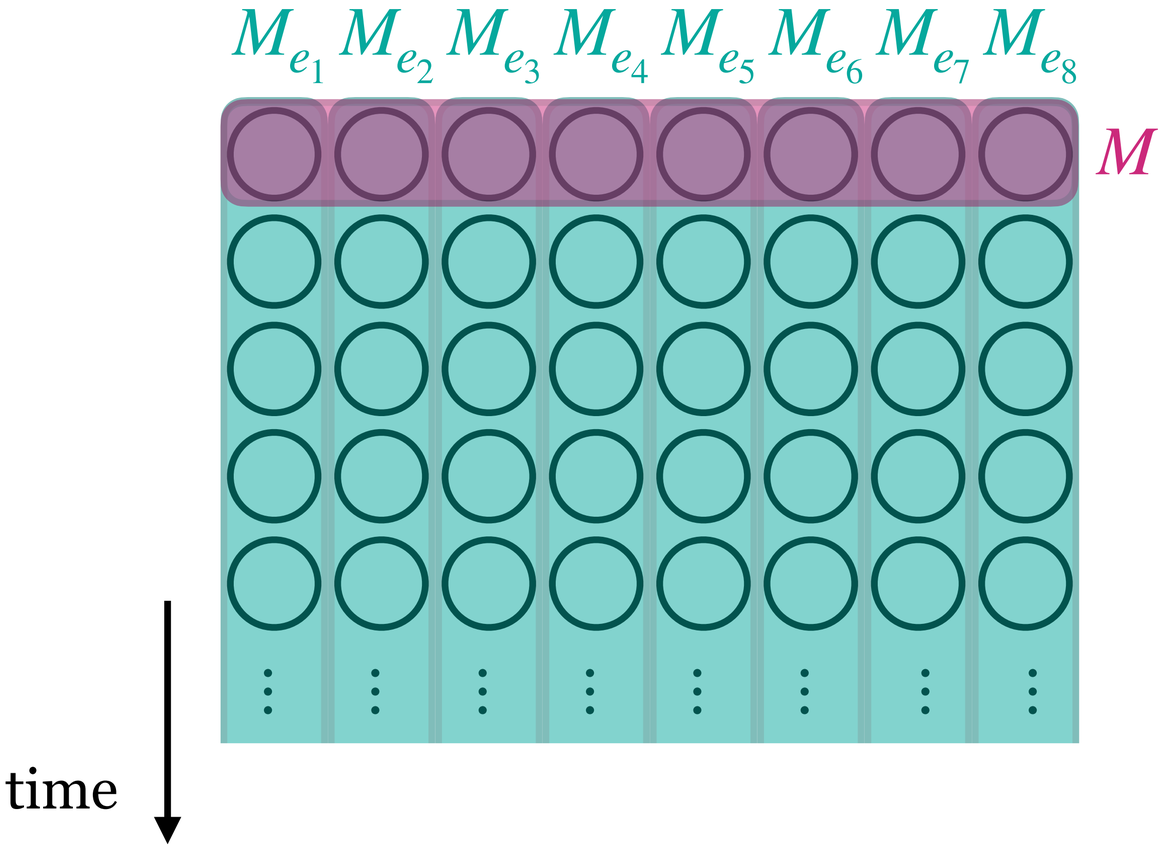

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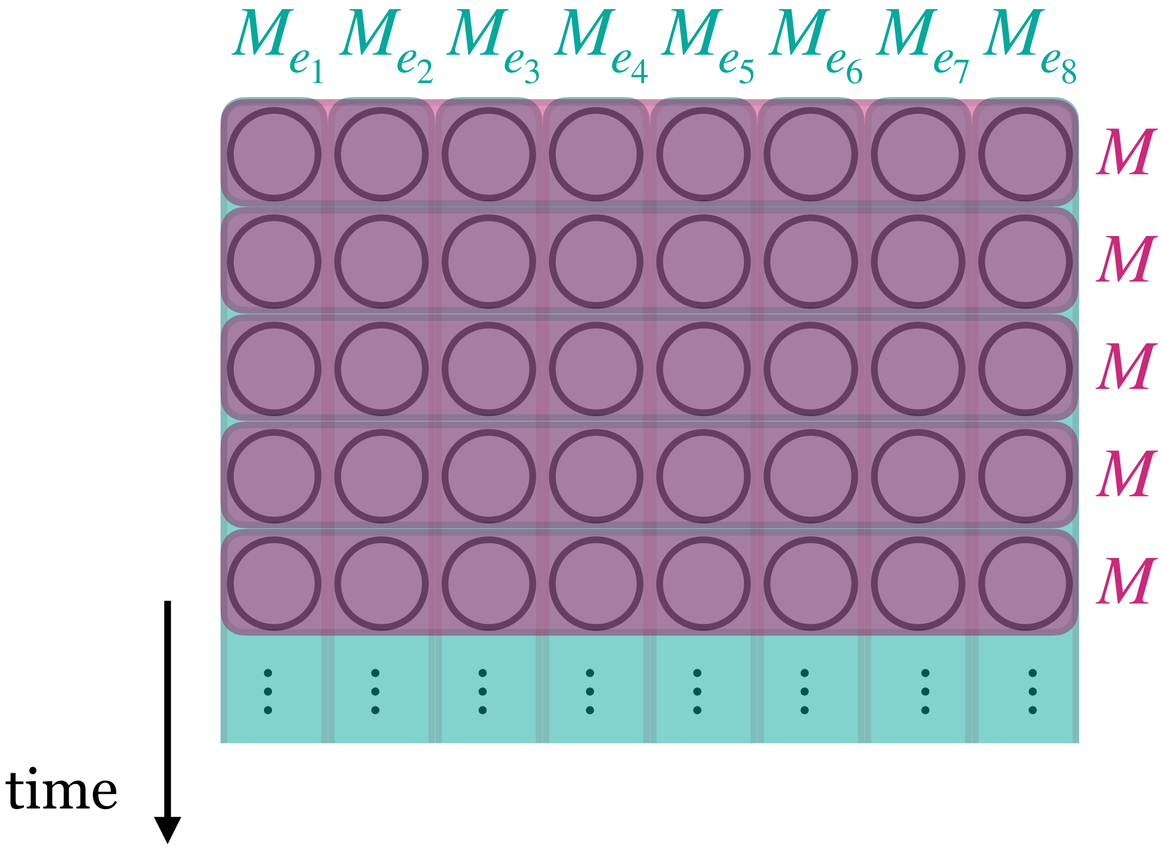

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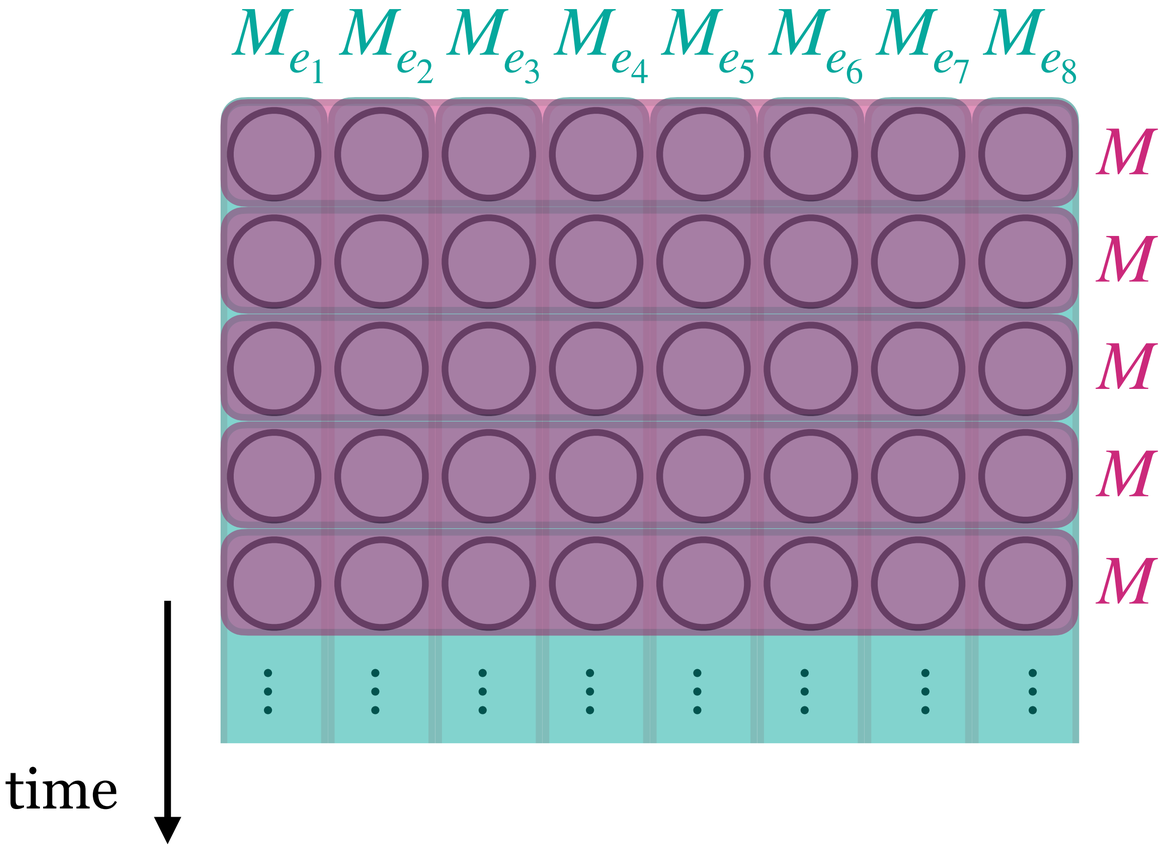

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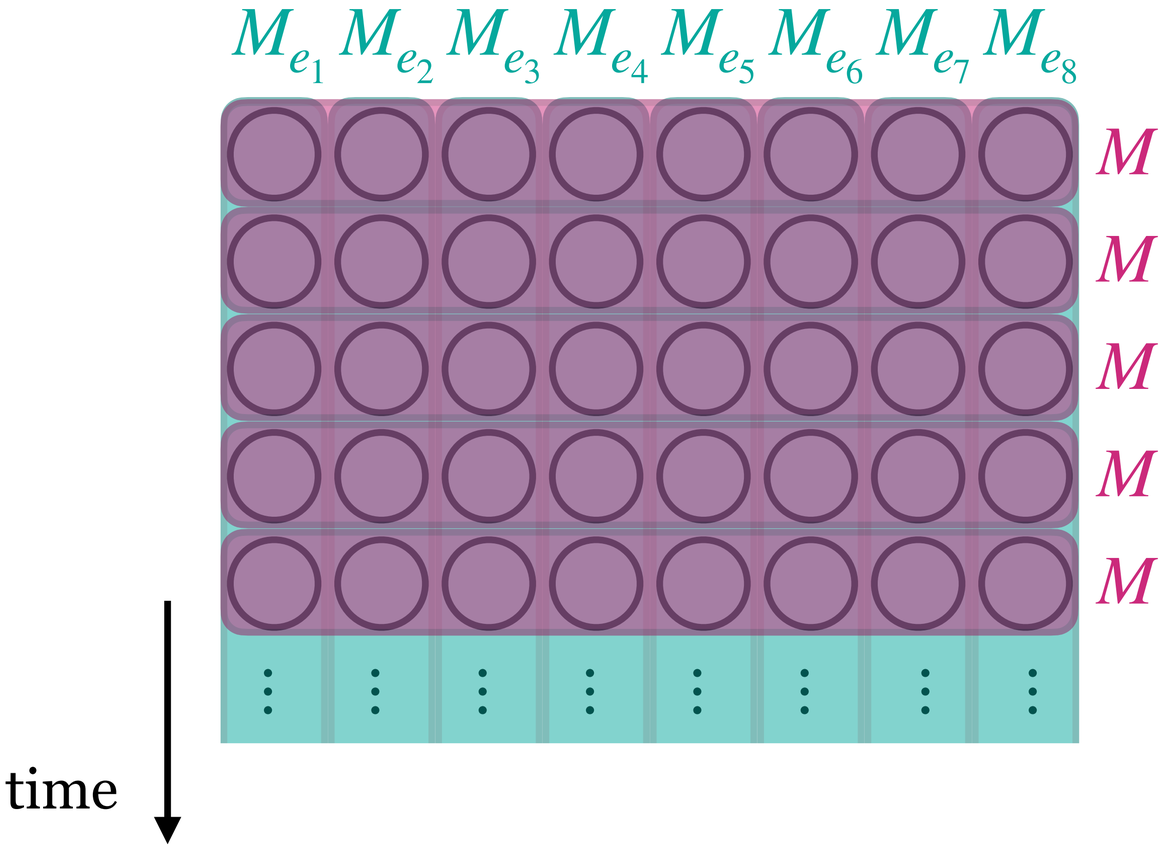

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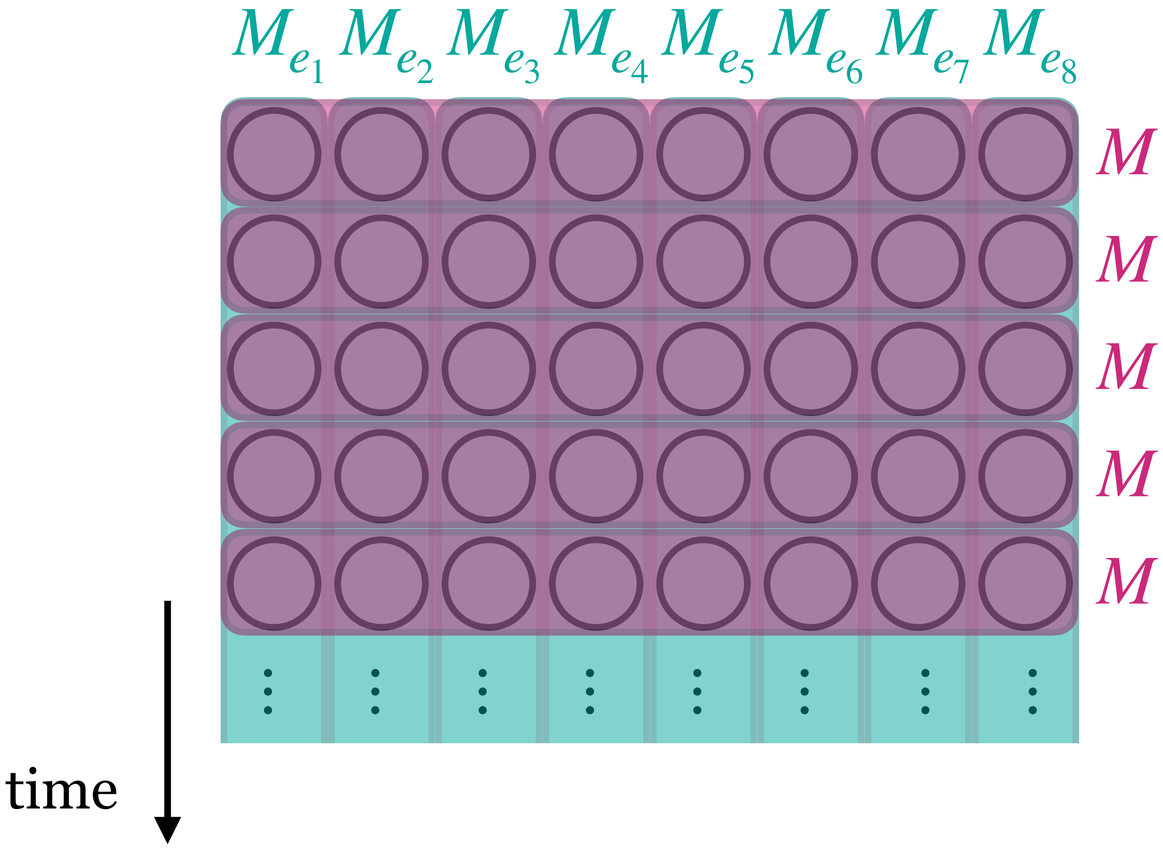

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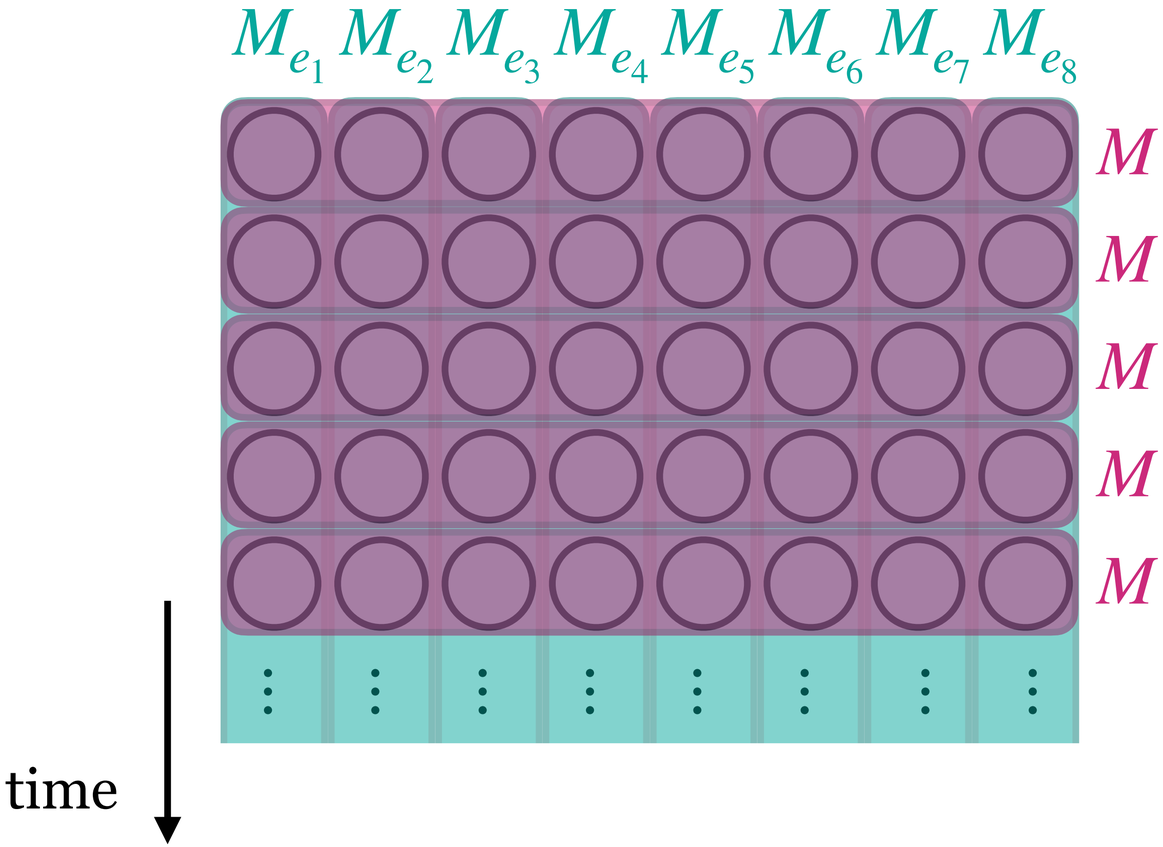

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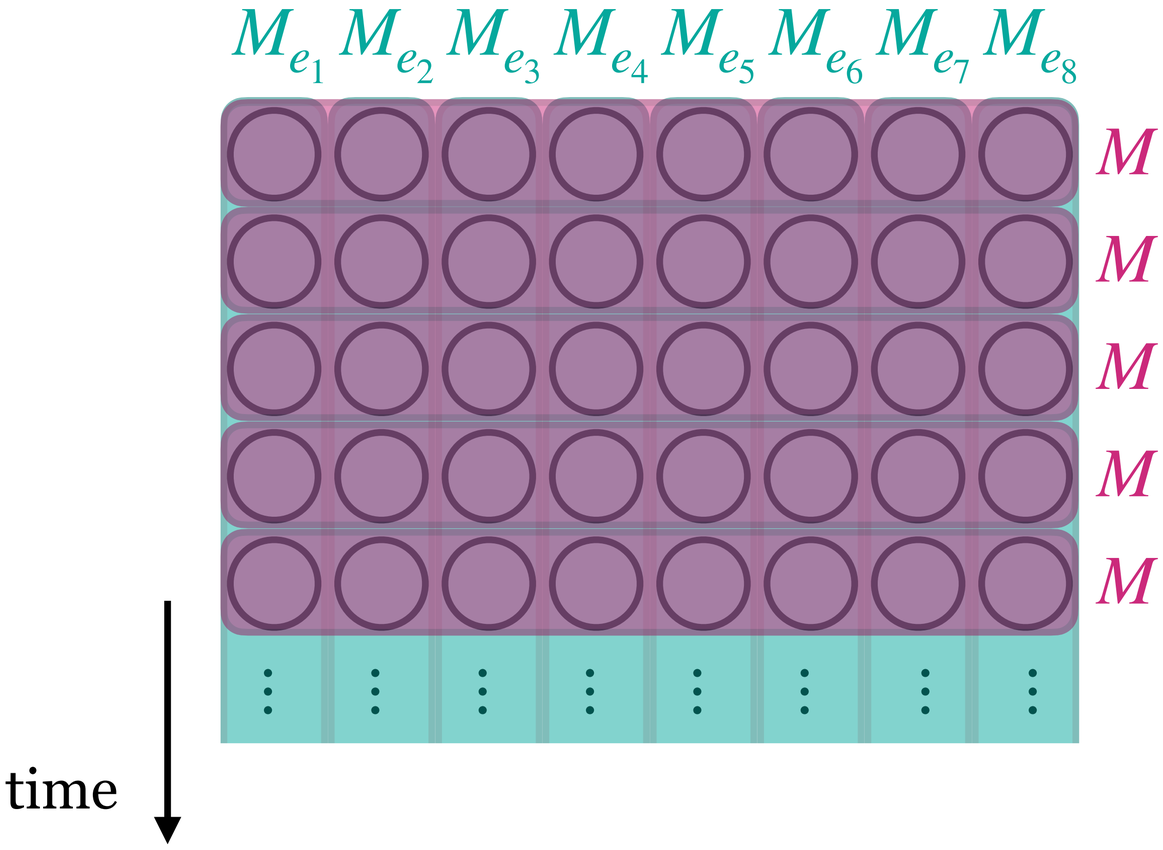

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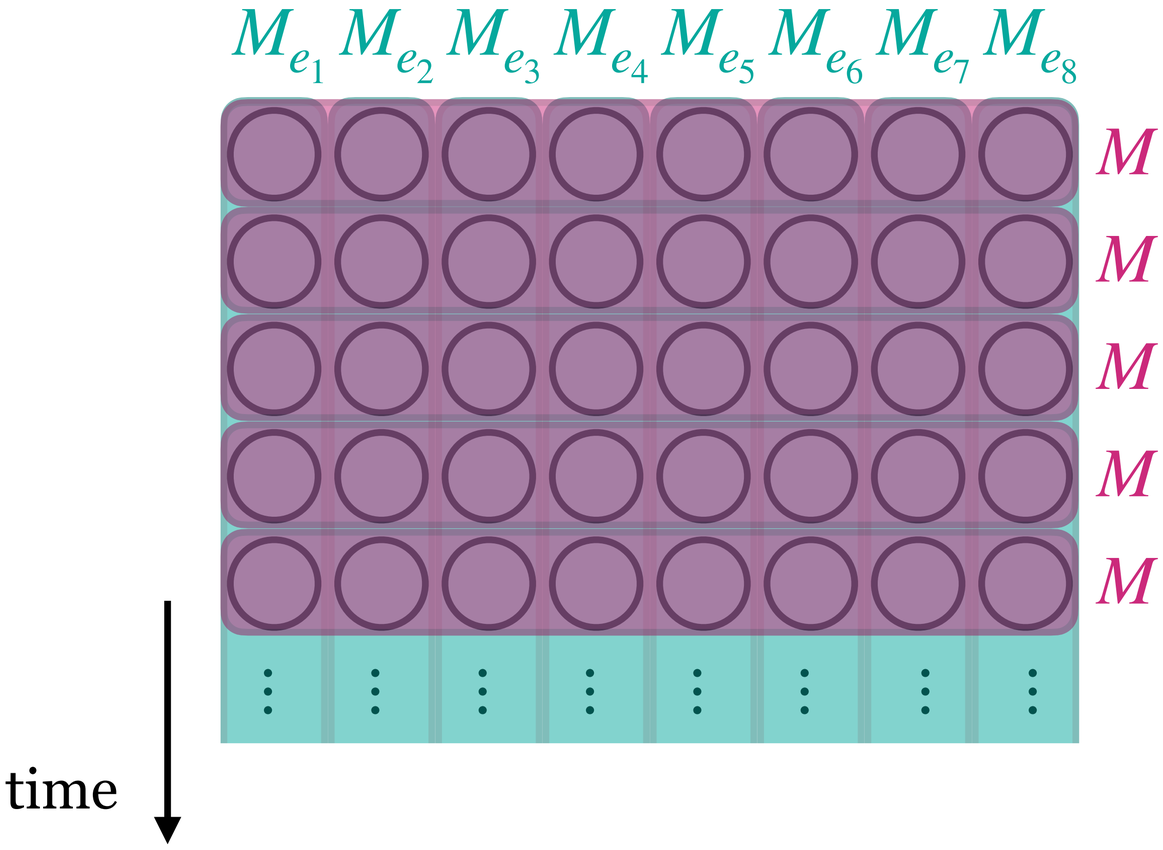

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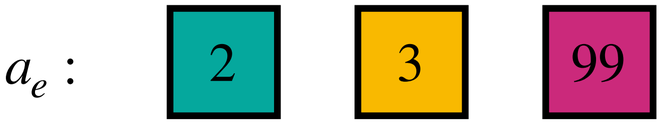

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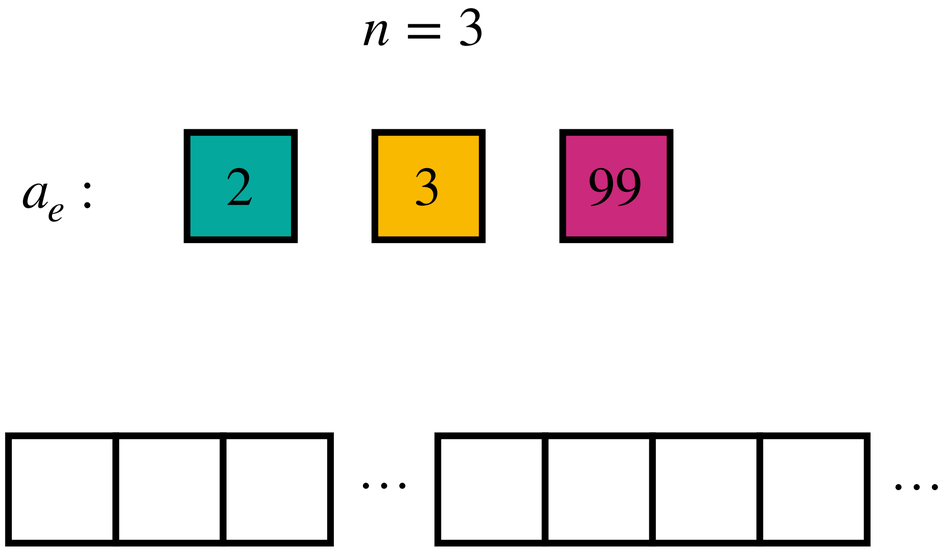

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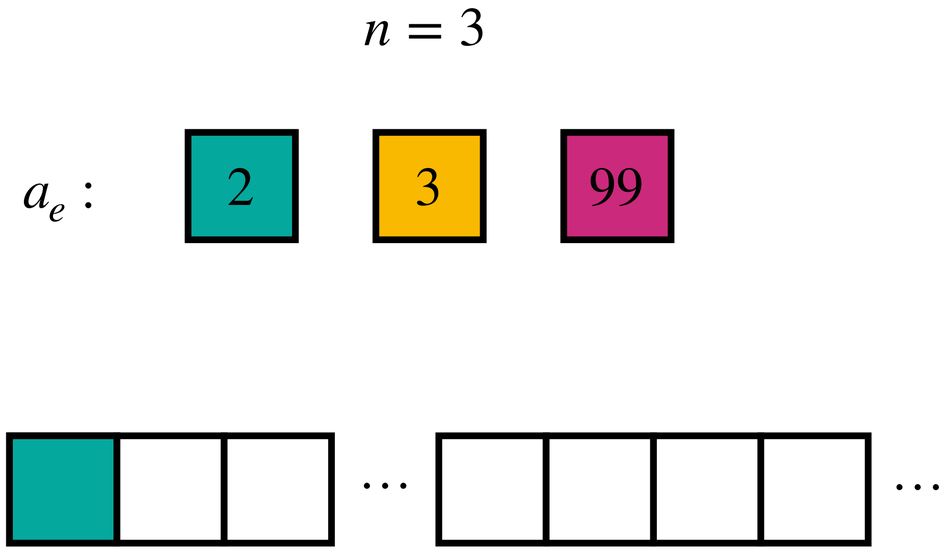

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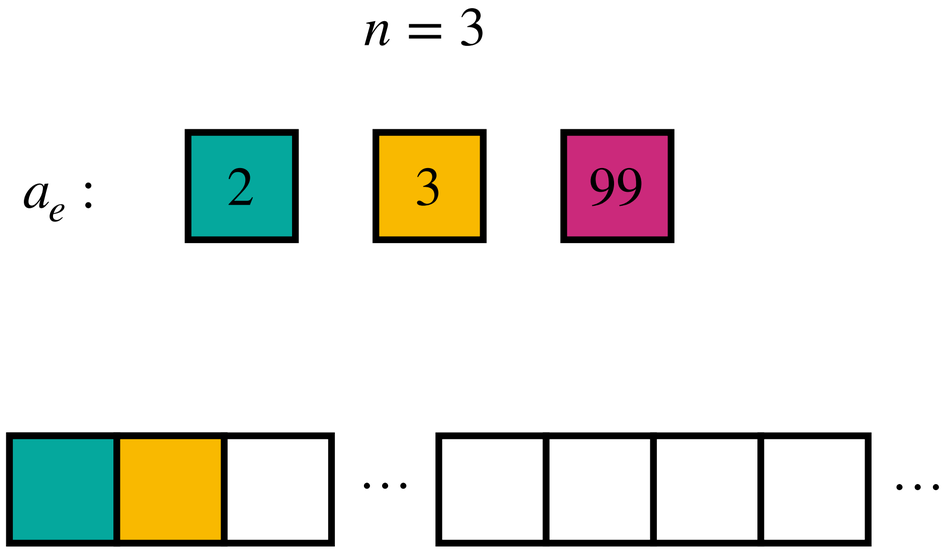

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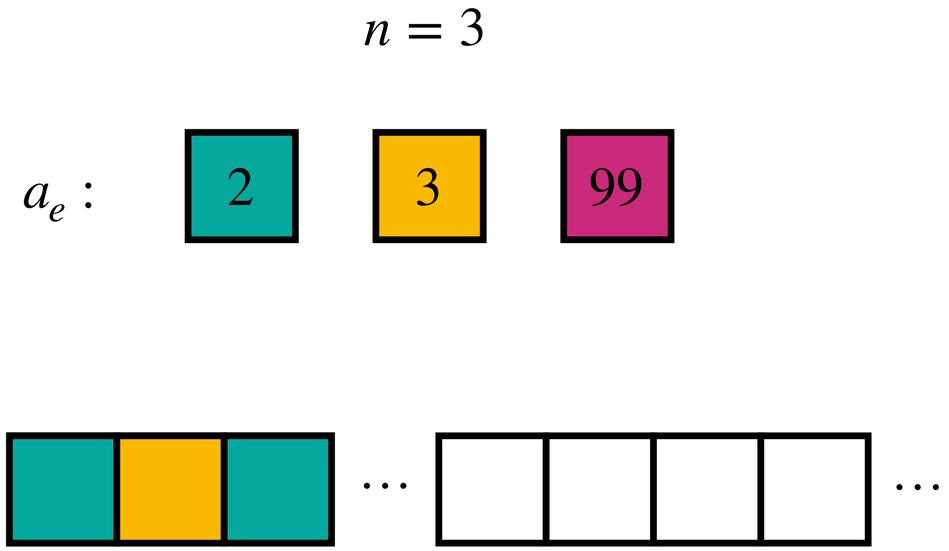

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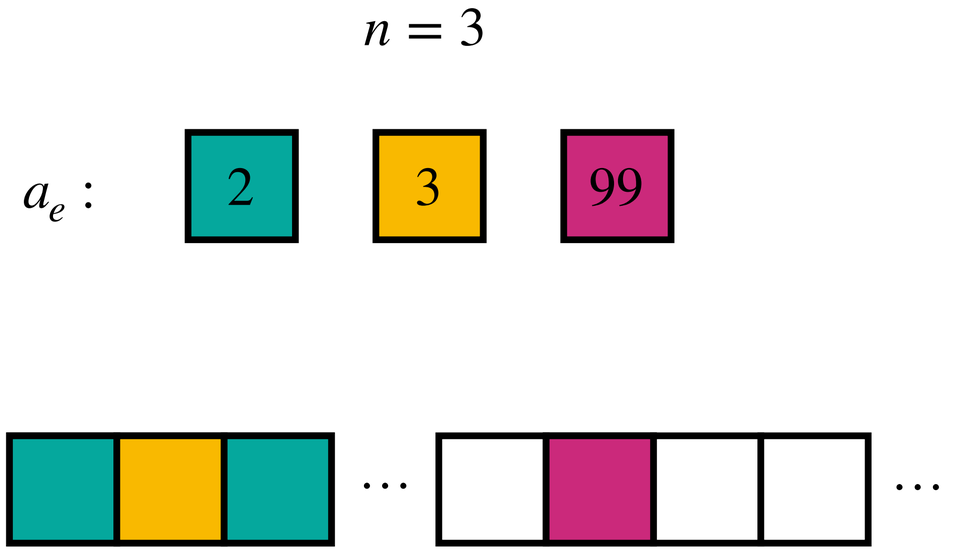

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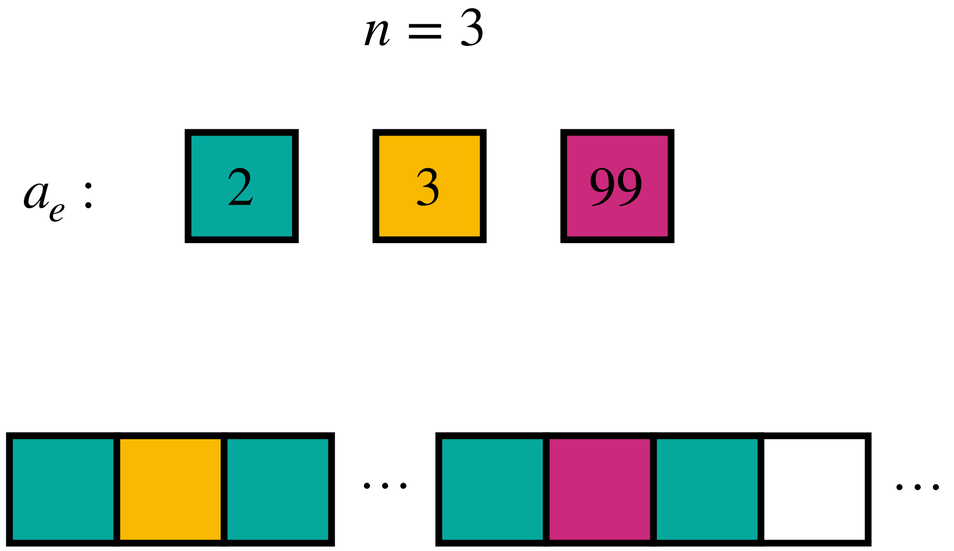

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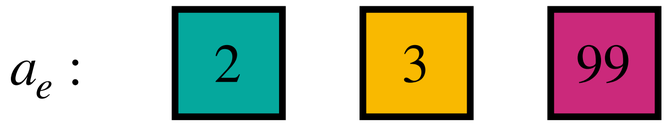

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- Was shown to be precisely 5/6 recently. [Kawamura, STOC'24]

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Thank You!