Scheduling with restrictions on job positions

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Agenda

- Problems $1|\beta, pd|\gamma$
- Problems $1|\beta, fp|\gamma$

pd (position dependent maintenance): After at most k jobs, the machine must be idle for at least s time units.

fp (fixed position): (must not be combined with preemption) For some jobs, the position within the job sequence is specified.

Definition $1|\beta, pd|\gamma$

- Single machine
- n jobs with processing time p_j and due-dates d_j (if applicable)
- β can be empty or contain (amongst others)
  - ready times r_j
  - preemption (pmtn)
- pd enforces maintenance operations
  - executed after changing *at most* k jobs (or partial jobs) on the machine
  - last for s units of time
  - are non-preemptive
- γ (objective function) e.g.
  - $\sum C_j$
  - $C_{\max} := \max C_j$
  - $L_{\max} := \max\{C_j - d_j\}$

(Fake) Applications

Light Bulb

Aircraft Engine & Landing Gear

Automotive Battery

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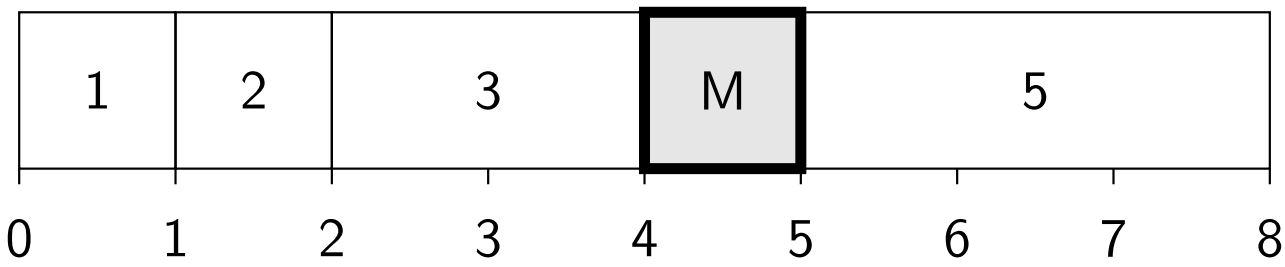
Example $1|pd|\sum C_j$

$k = 3, s = 1$

j	1	2	3	4
p_j	1	1	2	3

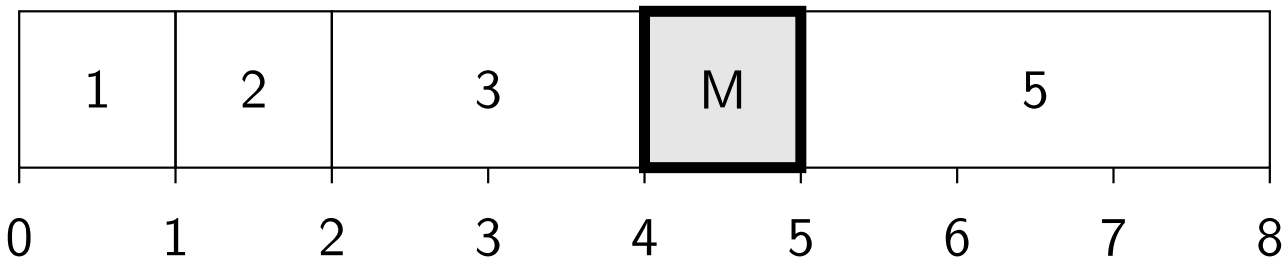


Objective function: $1 + 2 + 4 + 8 = 15$

Example $1|pd|C_{\max}$

$k = 3, s = 1$

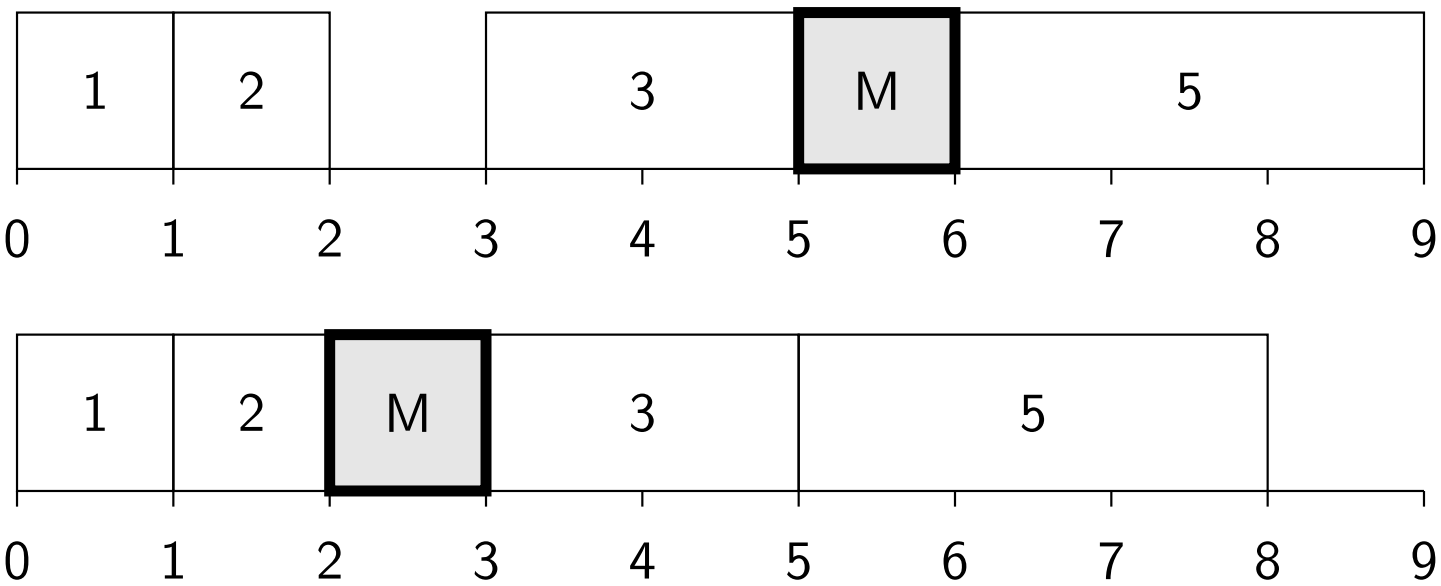
j	1	2	3	4
p_j	1	1	2	3



Example $1|r_j, pd|C_{\max}$

$k = 3, s = 1$

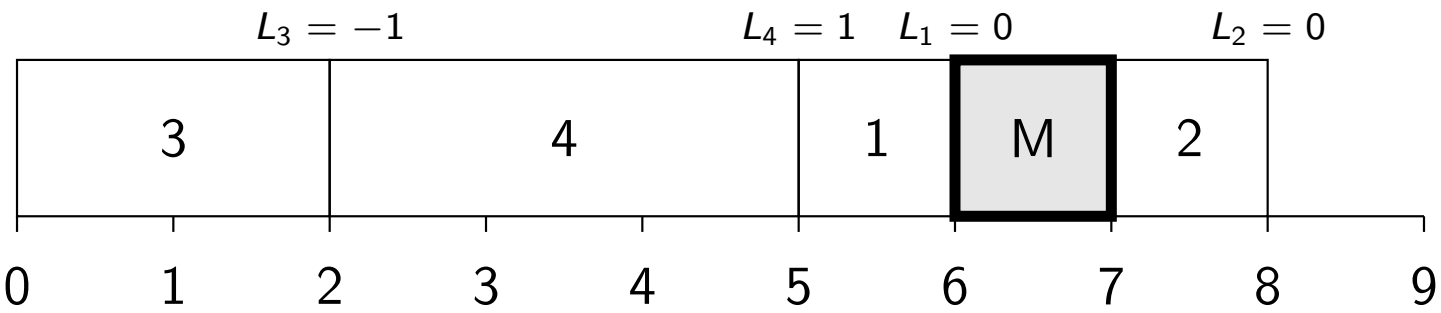
j	1	2	3	4
p_j	1	1	2	3
r_j	0	0	3	3



Example $1|pd|L_{\max}$

$k = 3, s = 1$

j	1	2	3	4
p_j	1	1	2	3
d_j	6	8	3	4



Objective function: $\max\{0, 0, -1, 1\} = 1$

Summary of the Examples

Property

1. $1|pd|\sum C_j$ *is solvable in* $O(n \log n)$
   (SPT-rule, maintenance operations as late as possible)
2. $1|pd|C_{\max}$ *is solvable in* $O(n)$
   (any order of jobs, maintenance operations as late as possible)
3. $1|r_j, pd|C_{\max}$ *is solvable in* $O(n^2)$
   (any order of jobs, DP for scheduling maintenance operations)
4. $1|pd|L_{\max}$ *is solvable in* $O(n \log n)$
   (EDD-rule, maintenance operations as late as possible)

Complexity results remain, even if preemption is allowed.

Is *pd* really boring?

$1|pmtn, pd|L_{\max}$ is easy,

$1|r_j, pmtn|L_{\max}$ is easy, but:

Property $(1|r_j, pmtn, pd|L_{\max})$

Problem $1|r_j, pmtn, pd|L_{\max}$ *is NP-hard in the strong sense.*

Proof.

Easy Proof: Setting $k = n$ and s very large allows us to "forbid" interruptions. $1|r_j|L_{\max}$ reduces to the problem at hand.

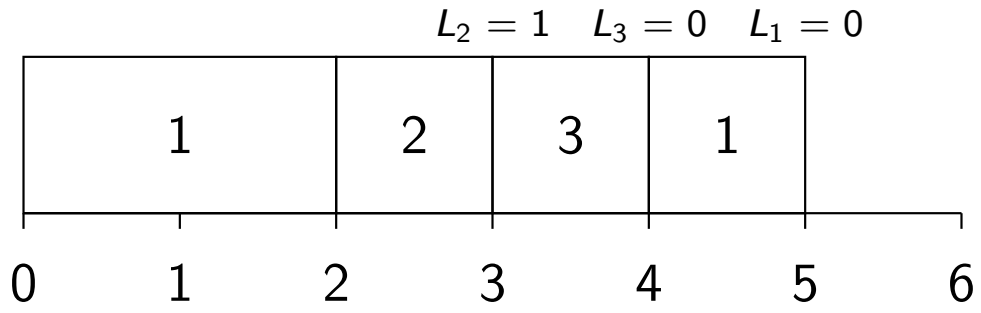
Another proof: shows that the problem is strongly NP-hard for any $k, s \in \mathbb{N}$.

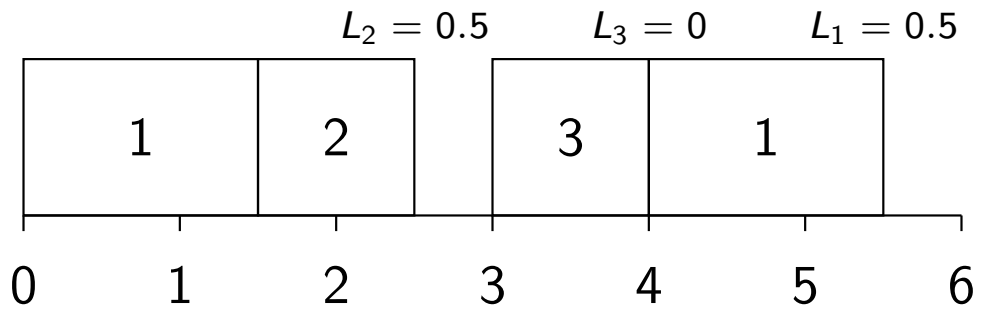
□

Surprising effects for $1|r_j, pmtn, pd|L_{\max}$

$k = 4, s = 1$

j	1	2	3
p_j	3	1	1
r_j	0	1	3
d_j	5	2	4





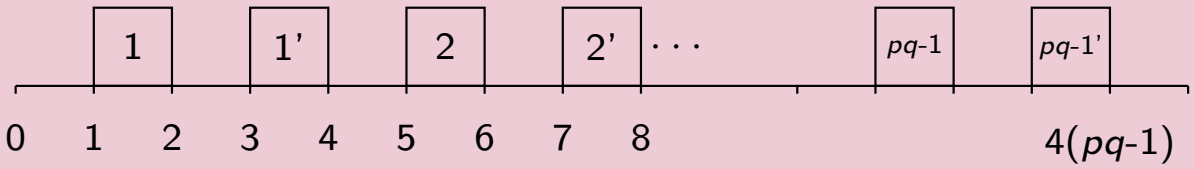
Fractional schedules for $1|r_j, pmtn, pd|L_{\max}$

Property

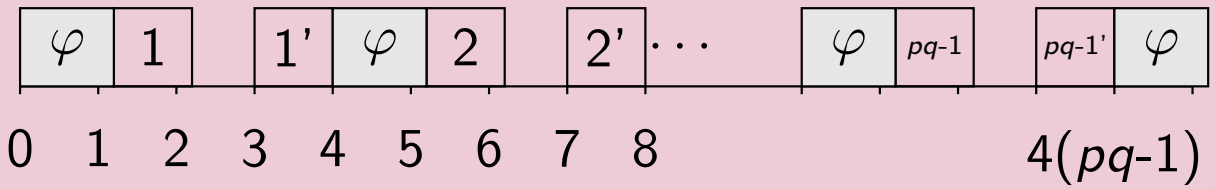
For every rational number $0 < \frac{p}{q} < 1$ *with* $p, q \in \mathbb{N}$*, there is an instance of* $1|r_j, pmtn, pd|L_{\max}$ *with optimal objective function value* $L_{\max} = \frac{p}{q}$.

Proof.

Set k such that there are at most $pq - 1$ interruptions and s large. There are $pq - 1$ pairs (!) of jobs with tight release dates and due dates.



Add another job φ with $r_\varphi = 0$, $d_\varphi = 4(pq - 1) + 1$ and $p_\varphi = pq + p^2$.

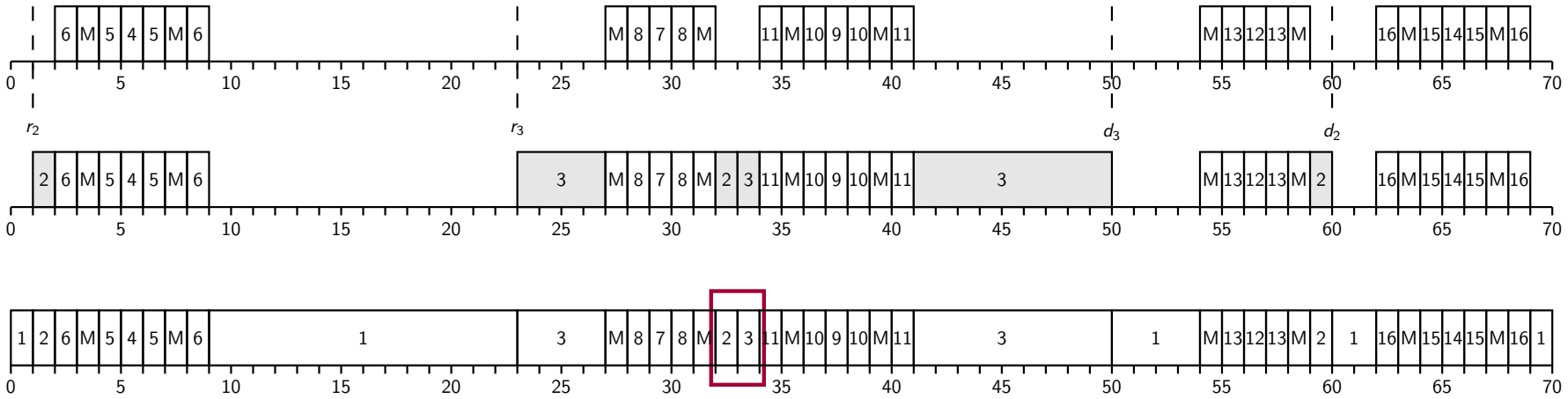


Number of interruptions for $1|r_j, pmtn, pd|L_{\max}$

From for $1|r_j, pmtn|L_{\max}$, we know that there are at most $n-1$ interruptions, each of which is induced by a release date.

Property

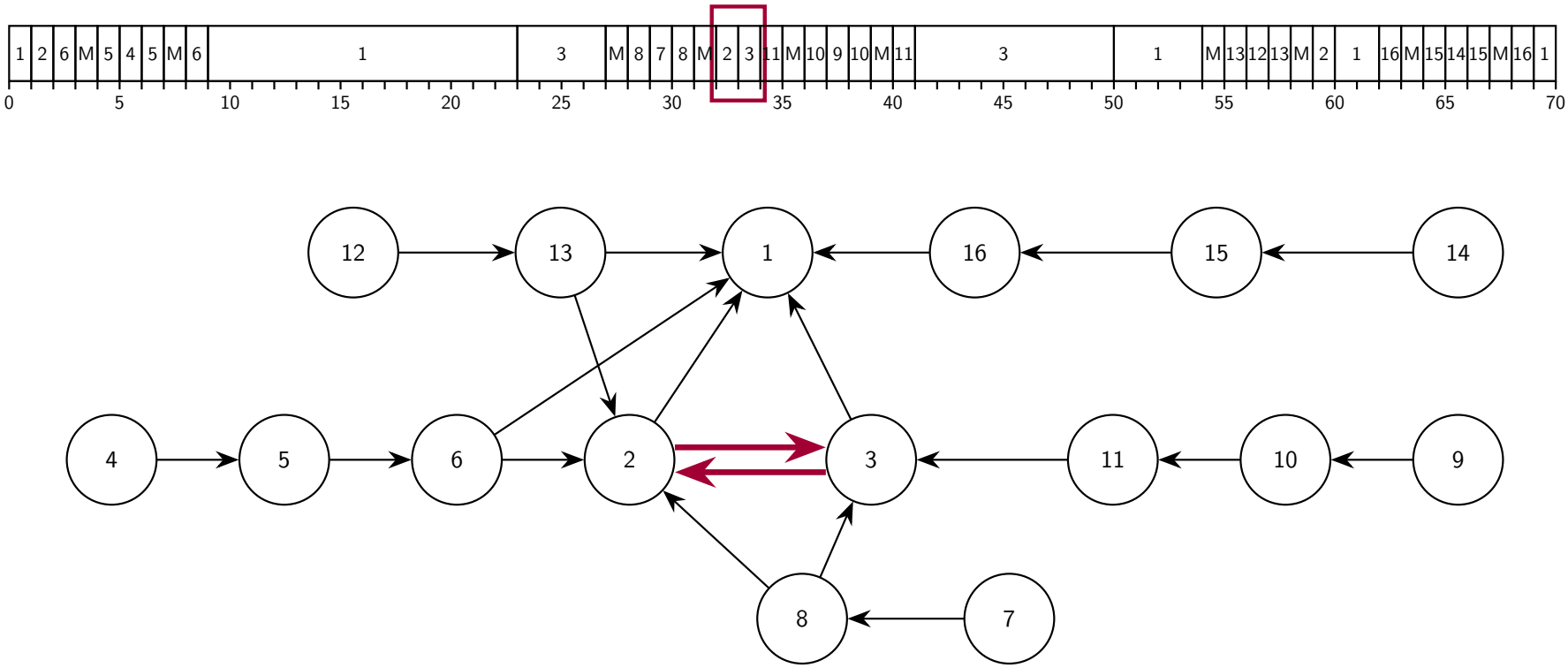
For any $\varepsilon > 0$, there is an instance with more than $\frac{13}{12}n - \varepsilon$ interruptions of jobs in any optimum schedule.



Number of interruptions for $1|r_j, pmtn, pd|L_{\max}$

Property

For any instance $1|r_j, pmtn, pd|L_{\max}$*, there is an optimum solution with at most* $2n-2$ *interruptions of jobs.*



Open questions in position dependent maintenance scheduling

- Tight bounds on the required number of interruptions in preemptive scheduling for various problem settings?
- Multi machine problems are not analyzed so far.

Definition $1|\beta, fp|\gamma$

- Single machine
- n jobs with processing time p_j and due-dates d_j (if applicable)
- fp means:
  - There is a set $Q \subset J$ of special jobs with $q := |Q| < n$.
  - The given injective function, $\eta : Q \to \{1, \ldots, n\}$, fixes the position of a special job in the schedule.
  - A special job j must be scheduled so that exactly $\eta(j)$ regular jobs are completed before j starts.
- γ (objective function) e.g.
  - $\sum C_j$
  - $\sum w_j C_j$
  - $L_{\max} := \max\{C_j - d_j\}$
  - $\sum_{j \in J} U_j$ with $U_j := 1$ if $C_j > d_j$ and $U_j := 0$ if $C_j \leq d_j$

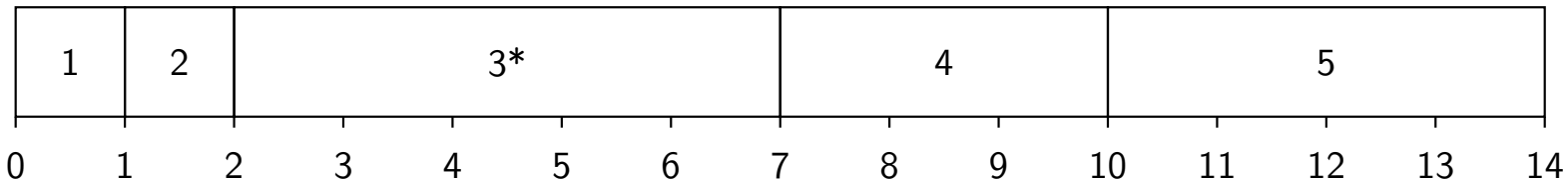
Please note that special jobs also contribute to the value of the objective function.

Applications

- Position dependent maintenance operations that are scheduled as late as possible and that contribute to the objective function value.
- Interim Acceptances in production scheduling.
- Disinfection jobs in Operating Room Scheduling

Example $1|fp|\sum C_j$

j	1	2	3*	4	5
p_j	1	1	5*	3	4



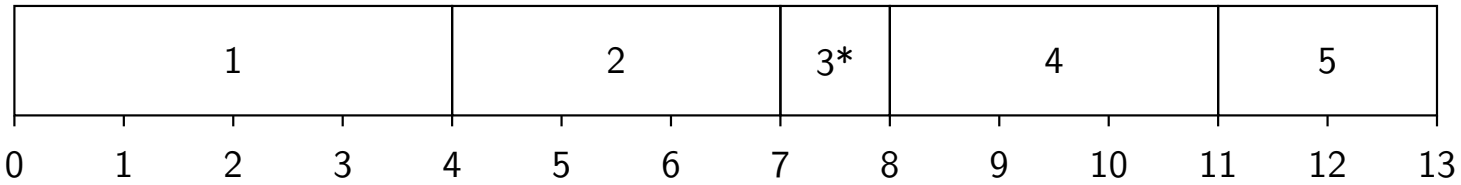
Objective function value: $1 + 2 + 7 + 10 + 14 = 34$

Property

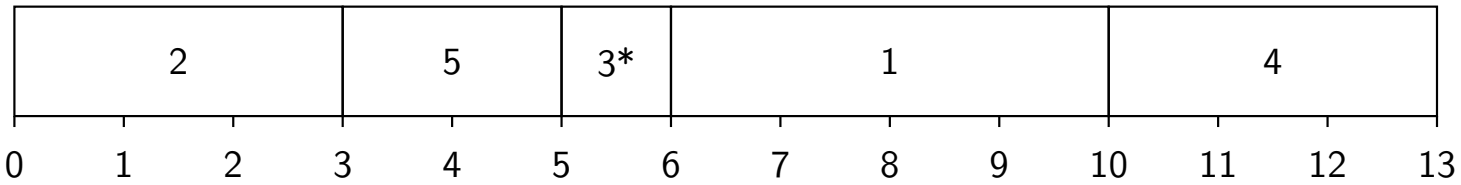
$1|fp|\sum C_j$ *is solvable in* $O(n \log n)$ *(SPT-rule).*

Example $1|fp|\sum w_j C_j$

j	1	2	3*	4	5
p_j	4	3	1*	3	2
w_j	5	3	9*	2	1



Objective function value (WSPT): $4 \cdot 5 + 7 \cdot 3 + 8 \cdot 9 + 11 \cdot 2 + 13 \cdot 1 = 148$



Objective function value (OPT): $10 \cdot 5 + 3 \cdot 3 + 6 \cdot 9 + 13 \cdot 2 + 5 \cdot 1 = 144$

Results for $1|fp|\sum w_j C_j$

Property

Regular jobs, which are not interrupted by special jobs, follow WSPT rule.

Whether or not $1|fp|\sum w_j C_j$ is NP-hard remains an open question, even when there is only one special job.

Property

There is a FPTAS, a $(1+\frac{\varepsilon}{n})$*-approx. with* $O\left(n^{3q+3}(\log W \log P)^{q+1}\left(\frac{1}{\varepsilon}\right)^{2q+2}\right)$ *runtime.*[1]

[1] Unpublished result by Kai Wang

Problem $1||L_{\max}$

Property

Problem $1||L_{\max}$ can optimally be solved using the earliest due date rule (EDD).

Example:

j	1	2	3	4	5	6
p_j	3	5	7	4	2	6
d_j	25	11	8	20	21	15
C_j	3	8	15	19	21	27
L_j	-22	-3	7	-1	0	$\underline{12}$

$\rightarrow$

j	3	2	6	4	5	1
p_j	7	5	6	4	2	3
d_j	8	11	15	20	21	25
C_j	7	12	18	22	24	27
L_j	-1	1	$\underline{3}$	2	$\underline{3}$	2

Problem $1|fp|L_{\max}$

Earliest due date rule is no longer optimal:

j	1	2	3*	4	5	6
p_j	3	5	7*	4	2	6
d_j	25	11	8*	20	21	15
C_j	3	8	15	19	21	27
L_j	-22	-3	7	-1	0	$\underline{12}$

$\xrightarrow{\text{EDD}}$

j	2	6	3*	4	5	1
p_j	5	6	7*	4	2	3
d_j	11	15	8*	20	21	25
C_j	5	11	18	22	24	27
L_j	-6	-4	$\underline{10}$	2	3	2

Optimal solution:

j	2	5	3*	6	4	1
p_j	5	2	7*	6	4	3
d_j	11	21	8*	15	20	25
C_j	5	7	14	20	24	27
L_j	-6	-14	$\underline{6}$	5	4	2

Problem $1|fp|L_{\max}$, Solution

Theorem

The DECISION VERSION of $1|fp|L_{\max}$ can be solved in $O(n^2)$.

So for a given threshold $L_{\max}$, we can decide whether there is a feasible solution. In fact, we can even assume (w.l.o.g.) that $L_{\max} = 0$ by changing due dates accordingly. If we, for example, ask for $L_{\max} = 6$:

j	1	2	3*	4	5	6
p_j	3	5	7*	4	2	6
d_j	25	11	8*	20	21	15

$\rightarrow$

j	1	2	3*	4	5	6
p_j	3	5	7*	4	2	6
d_j	31	17	14*	26	27	21

Problem $1|fp|L_{\max}$, Algorithm

1. Schedule jobs from the end (i.e., from time $P = \sum_{j \in J} p_j$) as follows.
2. If the next job is a special job, schedule this one and set $P = P - p_j$. If it is late, return NO.
3. Else, consider all jobs with $d_j \geq P$. If there are none, return NO. Schedule the longest job j of these jobs and set $P = P - p_j$.
4. Repeat steps 2 and 3 until $P = 0$. If so, return YES.

Example for $L_{\max} = 6$ ($P = 27$):

j	1	2	3*	4	5	6
p_j	3	5	7*	4	2	6
d_j	31	17	14*	26	27	21

$\overset{P=24}{\rightarrow}$

j	2	4	3*	5	6	1
p_j	5	4	7*	2	6	3
d_j	17	26	14*	27	21	31

$P = 20$:

j	2	5	3*	6	4	1
p_j	5	2	7*	6	4	3
d_j	17	27	14*	21	26	31

$\overset{P=14}{\rightarrow}$

j	2	5	3*	6	4	1
p_j	5	2	7*	6	4	3
d_j	17	27	14*	21	26	31

and so on

Problem $1|fp|L_{\max}$, Solution

Theorem

$1|fp|L_{\max}$ *can be solved in* $O(n^2 \log n)$.

Property

Let $P := \sum p_j$ *and* $\underline{L}_{\max} \leq L^*_{\max}$ *be a lower bound. If* $P - d_j < \underline{L}_{\max}$ *holds for a regular job* j*, then job* j *does not induce the objective function value in any (non-idle) solution.*

Conclusion: If the decision problem returns NO for threshold $P - d_j$, i.e., $P - d_j \leq \underline{L}_{\max}$, j can be scheduled in any position. Collect these jobs in set J'.

Property

If $P - d_j > \overline{L}_{\max}$ *holds for a regular job* j *and an upper bound* $\overline{L}_{\max}$*, then job* j *must not be scheduled last in an optimal solution*

Conclusion: If the decision problem returns YES for threshold $P - d_j$, job j must not be scheduled last, but a job from set J'.

Problem $1|fp|L_{\max}$, Algorithm

1. Use lower and upper bounds. Sort regular jobs in non-increasing order of d_j.
2. (Special jobs are scheduled on their intended position.)
3. For $j = 1, 2, \ldots$: check feasibility for $P - d_j$ ($P = \sum p_j$) until job k returns YES. Update bounds and assign $1, \ldots, k-1$ to J'.
4. Schedule the longest job from J' last and call the algorithm with $n - 1$ jobs.

$\underline{L}_{\max} = 3$ (optimal $1||L_{\max}$-solution) $\quad \overline{L}_{\max} = 10$ (EDD-solution) $\quad P = 27$

j	1	5	3*	4	6	2
p_j	3	2	7*	4	6	5
d_j	25	21	8*	20	15	11

$P - d_1 = 2 \leq \underline{L}_{\max} \rightarrow J' = \{1\}$

$P - d_5 = 6 \rightarrow$ YES $\quad$ Job 1 is scheduled last.

Problem $1|fp|L_{\max}$, Algorithm

1. Use lower and upper bounds. Sort regular jobs in non-increasing order of d_j.
2. (Special jobs are scheduled on their intended position.)
3. For $j = 1, 2, \ldots$: check feasibility for $P - d_j$ ($P = \sum p_j$) until job k returns YES. Update bounds and assign $1, \ldots, k-1$ to J'.
4. Schedule the longest job from J' last and call the algorithm with $n - 1$ jobs.

$\underline{L}_{\max} = 3$ $\quad\quad$ $\overline{L}_{\max} = 6$ (Update) $\quad\quad$ $P = 24$

j	5	4	3*	6	2	1
p_j	2	4	7*	6	5	3
d_j	21	20	8*	15	11	25

$P - d_5 = 3 \leq \underline{L}_{\max} \rightarrow J' = \{5\}$

$P - d_4 = 4$ delivers NO $\rightarrow J' = \{4, 5\}$

$P - d_6 = 9 \rightarrow$ YES $\quad\quad$ Job 4 is scheduled last.

Problem $1|fp|L_{\max}$, Algorithm

1. Use lower and upper bounds. Sort regular jobs in non-increasing order of d_j.
2. (Special jobs are scheduled on their intended position.)
3. For $j = 1, 2, \ldots$: check feasibility for $P - d_j$ ($P = \sum p_j$) until job k returns YES. Update bounds and assign $1, \ldots, k-1$ to J'.
4. Schedule the longest job from J' last and call the algorithm with $n - 1$ jobs.

$\underline{L}_{\max} = 4$ $\quad$ $\overline{L}_{\max} = 6$ $\quad$ $P = 20$

j	5	6	3*	2	4	1
p_j	2	6	7*	5	4	3
d_j	21	15	8*	11	20	25

Keep $J' = \{5\}$

$P - d_6 = 5$ delivers NO. Update of the bounds shows: optimal solution found

Summary of the Results

Position-dependent maintenance (pd)

Problems in P: $1|pd|\sum C_j$ $1|r_j, pd|C_{\max}$ $1|pd|L_{\max}$

$1|pmtn, pd|\sum C_j$ $1|pmtn, r_j, pd|C_{\max}$ $1|pmtn, pd|L_{\max}$

NP-hard: $1|pmtn, r_j, pd|L_{\max}$

Fixed-position constrained (fp)

Problems in P: $1|fp|\sum C_j$ $1|fp|L_{\max}$

Open problems: $1|fp|\sum w_j C_j$ $1|fp|\sum U_j$

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APPENDIX: Problem $1||\sum U_j$

Property

Problem $1||\sum U_j$ can optimally be solved using Moore's algorithm.

1. Initialization: Given an empty set of rejected jobs R.
2. Stop: Schedule all non-rejected (regular) jobs according to earliest due dates. If all non-rejected (regular) jobs are on-time, add all jobs from R to the end of the schedule and stop.
3. Reject job: Let j be the earliest tardy (regular) job, and let i be a (regular) job with longest processing time among the first j jobs of the schedule. Add i to the set of rejected jobs R and go to Step 2.

j	3	2	6	4	5	1
p_j	7	5	6	4	2	3
d_j	8	11	15	20	21	25
C_j	7	12	18	22	24	27
L_j	-1	1	3	2	3	2

$\rightarrow$

j	2	6	4	5	1	3
p_j	5	6	4	2	3	7
d_j	11	15	20	21	25	8
C_j	5	11	15	17	20	27
L_j	-6	-4	-5	-4	-5	19

Problem $1|fp|\sum U_j$, Example

Moore's algorithm is no longer optimal:

j	1	2	3*	4	5	6
p_j	3	5	7*	4	2	6
d_j	25	11	8*	20	21	15
C_j	3	8	15	19	21	27
L_j	-22	-3	$\underline{7}$	-1	0	12

W.l.o.g. we assume that each schedule starts and ends with dummy special jobs of zero length and maximal due date.

Problem $1|fp|\sum U_j$, Property 1

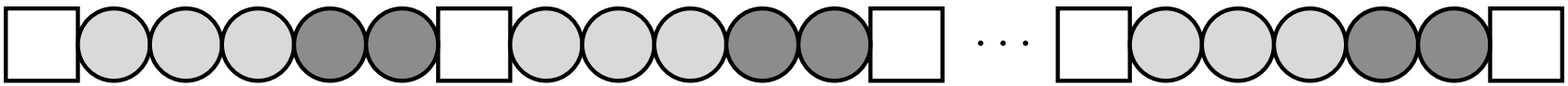
Property

There is always an optimal solution so that between any two special jobs all on-time jobs precede the tardy jobs.

Proof.

Exchange jobs in the "wrong" order. □

Dark grey: tardy jobs
Light grey: on-time job
no filling: tardiness unknown



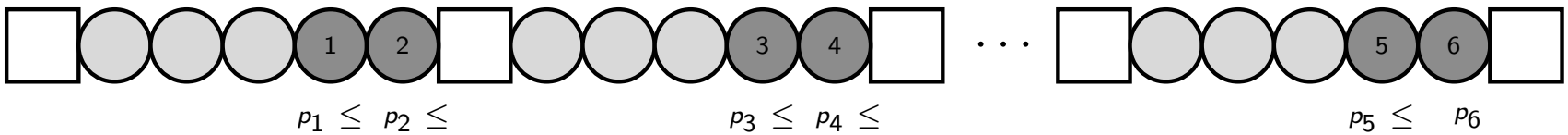
Problem $1|fp|\sum U_j$, Property 2

Property

There is always an optimal solution so that tardy regular jobs are sorted non-decreasingly with respect to processing times.

Proof.

Exchange jobs in the "wrong" order. □



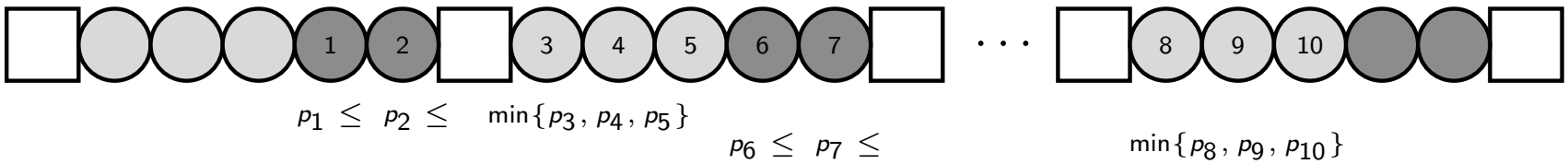
Problem $1|fp|\sum U_j$, Property 3

Property

There is always an optimal solution so that a tardy regular has no longer processing time than all regular on-time jobs that follow in the schedule.

Proof.

Exchange jobs in the "wrong" order. □



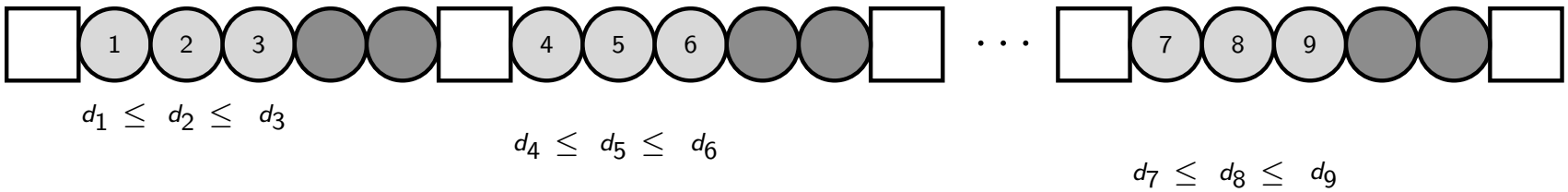
Problem $1|fp|\sum U_j$, Property 4

Property

There is always an optimal solution so that regular on-time jobs, which are not separated by a special job, are sorted according to EDD.

Proof.

Exchange jobs in the "wrong" order. □



Problem $1|fp|\sum U_j$, Definition

Definition

For a regular job $i \in J \backslash Q$ let

$$S[i] := \{j \in J \backslash Q | d_i \leq d_j, p_i > p_j\}$$

be the set of jobs that have no smaller due date but a smaller processing time and let

$$L[i] := \{j \in J \backslash Q | d_i \leq d_j, p_i \leq p_j\}.$$

Note that in Moore's algorithm, a job i is only on-time if all jobs from $S[i]$ are on-time.
Moreover, if the job with smallest due date i and all jobs from $S[i]$ can be scheduled on time, then Moore's algorithm schedules job i first and on-time.

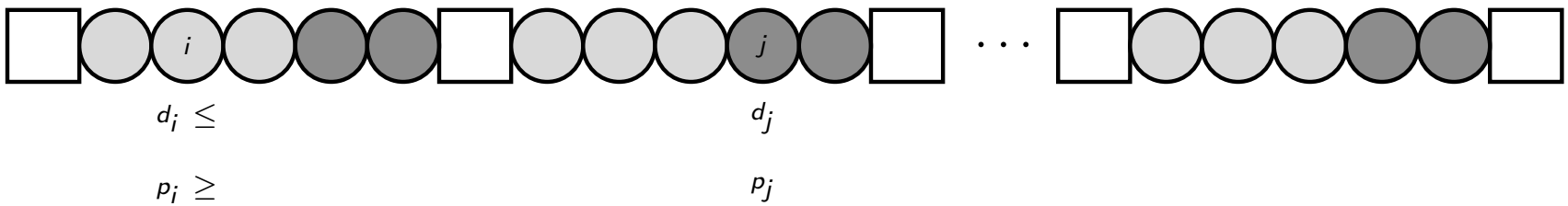
Problem $1|fp|\sum U_j$, Property 5

Property

There is always an optimal solution so that every regular job $i \in J \setminus Q$ is on-time, only if each job $j \in S[i]$ is on-time.

Proof.

Assume for a contradiction that there are only optimal solutions in which job i is on-time but some job $j \in S[i]$ is tardy. As $d_i \leq d_j$, this can only happen if $i \preceq_\sigma j$. However, then we exchange the position of both jobs so that job j is on-time. As $p_j < p_i$, jobs in between the two only advance in time so that the overall objective function does not worsen. This contradicts our assumption. □



Problem $1|fp|\sum U_j$, Example Property 5

If i and $S[i]$ can all be scheduled on-time in problem $1|fp|\sum U_j$, is i then on-time in some optimal solution? NO!

j	1	2	3*	4	5	6	7
p_j	4	2	1*	3	3	5	5
d_j	4	19	7*	10	13	12	17
C_j	4	6	7	10	13	18	23

- Job 1 and $S[1] = \{2, 4, 5\}$ can all be scheduled on-time.
- If Job 1 is on time, there are at least two tardy jobs (note that the last job is always tardy).
- If Job 1 is scheduled last, we obtain a solution with just one tardy job.

j	4	5	3*	6	7	2	1
p_j	3	3	1*	5	5	2	4
d_j	10	13	7*	12	17	19	4
C_j	3	6	7	12	17	19	23

Problem $1|fp|\sum U_j$, What now?

So how can we solve $1|fp|\sum U_j$? I have no idea! :,-(

Let us consider a special case with just one special job $1|fp, q = 1|\sum U_j$.

Property

If there is just one special job, $1|fp|\sum U_j$ can be solved in $O(n^5)$.