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Rutgers Business School
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# Min sum ordering problems with applications to scheduling


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# Plan

- A general min sum ordering problem
- A 2-approximation for a subclass of problems:

Fokkink, R., Lidbetter, T. and Végh, L.A., 2019. On submodular search and machine scheduling. *Mathematics of Operations Research*, *44*(4), pp.1431-1449.

- A $4\alpha$-approximation for another subclass:

Happach, F., Hellerstein, L. and Lidbetter, T., 2022. A general framework for approximating min sum ordering problems. *INFORMS Journal on Computing*, *34*(3), pp.1437-1452.

# A classic scheduling problem

- There are $n$ jobs that must be scheduled
- The weight (importance) of job $i$ is $w_i$
- The processing time of job $i$ is $p_i$

| | | | | | |
|---|---|---|---|---|---|
| $t_i$: | 3 hrs | 7 hrs | 3 hrs | 2 hrs | 6 hrs |
| $w_i$: | $2/15$ | $5/15$ | $3/15$ | $1/15$ | $4/15$ |

*Problem: In what order should the jobs be scheduled to minimize the sum of the weighted completion times? Usually denoted* $1||\sum w_i C_i$.

*Solution: Perform the jobs in non-increasing order of the index* $w_i/t_i$.

*(Smith, 1956)*

"Smith's rule"

# Writing in a different form

For a particular schedule, let $S_j$ = set of jobs up to and including job $j$. Then

Sum of weighted completion times = $\sum w_j C_j = \sum \left( g(S_j) - g(S_j - \{j\}) \right) f(S_j)$,

where $f(S) = \sum_{j \in S} p_j$ and $g(S) = \sum_{j \in S} w_j$.

Note: the functions $f$ and $g$ are both non-decreasing and *modular:*

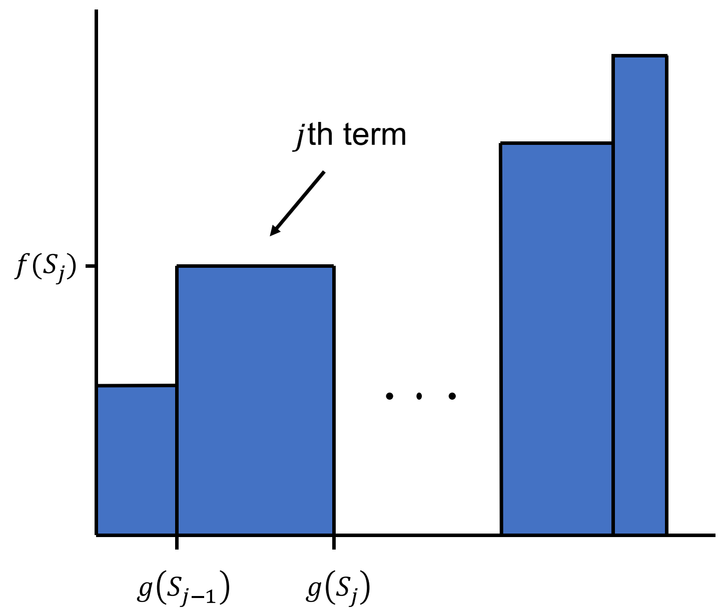
$f(S) = \sum_{j \in S} f(j)$ and $g(S) = \sum_{j \in S} g(j)$.

# Problem definition

- Finite set $V$ of cardinality $n$
- Non-decreasing cost function $f\colon \mathcal{F} \to \mathbb{R}$, where $f(\emptyset) = 0$.
- Non-decreasing weight function $g\colon \mathcal{F} \to \mathbb{R}$, where $g(\emptyset) = 0$.
- Minimize

$$\sum_{j=1}^{n} f(S_j)\left(g(S_j) - g(S_{j-1})\right),$$

over *chains* $\mathcal{S} \equiv (S_0, S_1, \ldots, S_n)$ satisfying
$\emptyset = S_0 \subset S_1 \cdots \subset S_n = V$, $|S_i| = i$ for all $i$.



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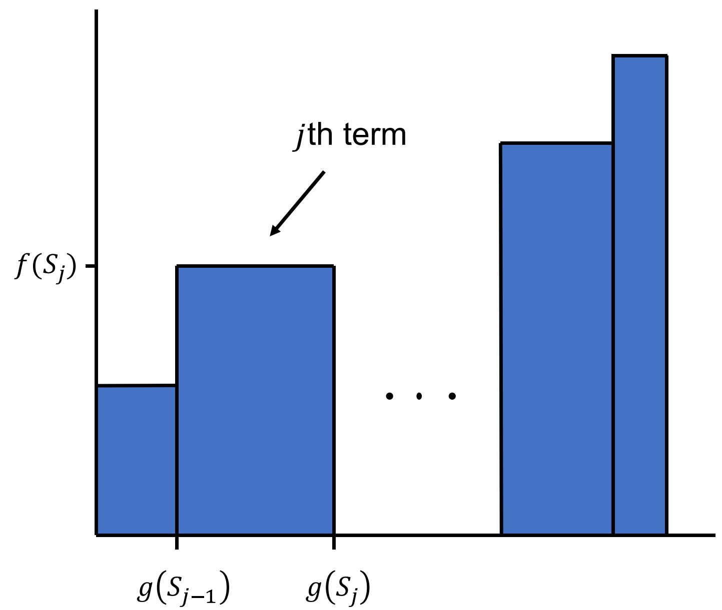
- Non-decreasing weight function $g\colon \mathcal{F} \to \mathbb{R}$, where $g(\emptyset) = 0$.
- Min-sum ordering problem (MSOP): minimize

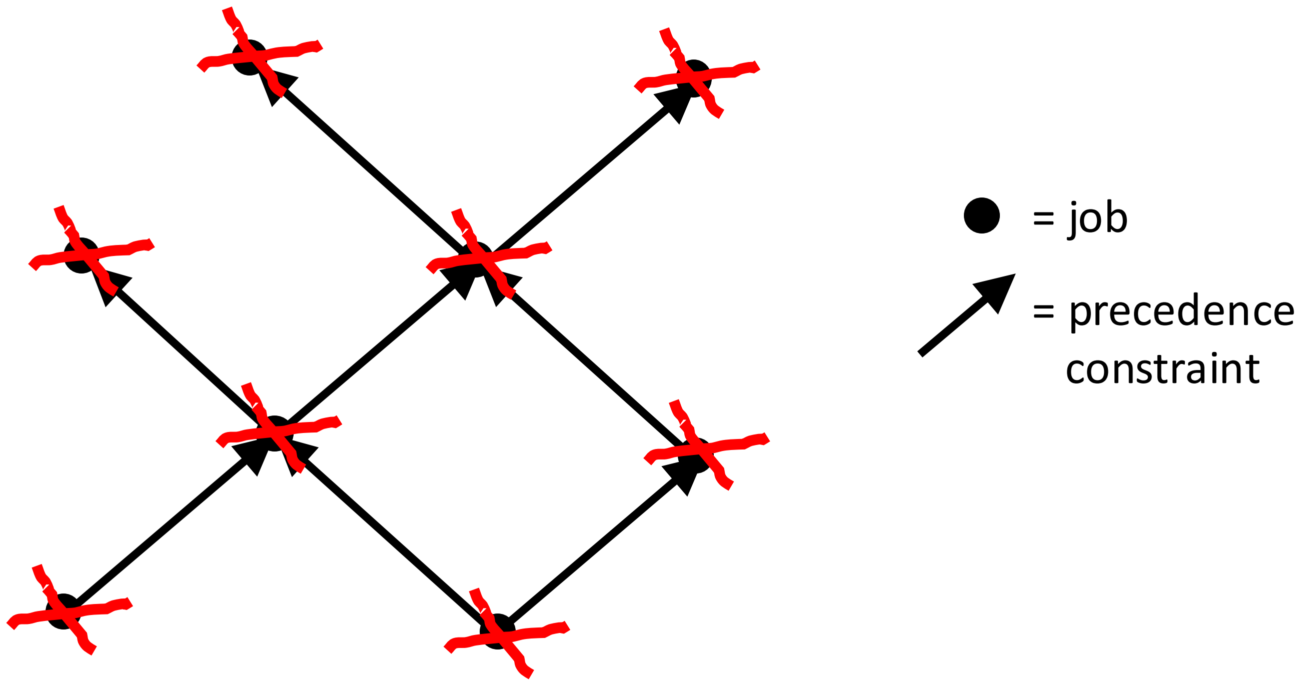
$$\sum_{j=1}^{k} f(S_j)\left(g(S_j) - g(S_{j-1})\right),$$

over *chains* $\mathcal{S} \equiv (S_0, S_1, \dots, S_k)$ satisfying $\emptyset = S_0 \subset S_1 \cdots \subset S_k = V$.



# A harder scheduling problem

Precedence constraints given by a DAG (directed acyclic graph).

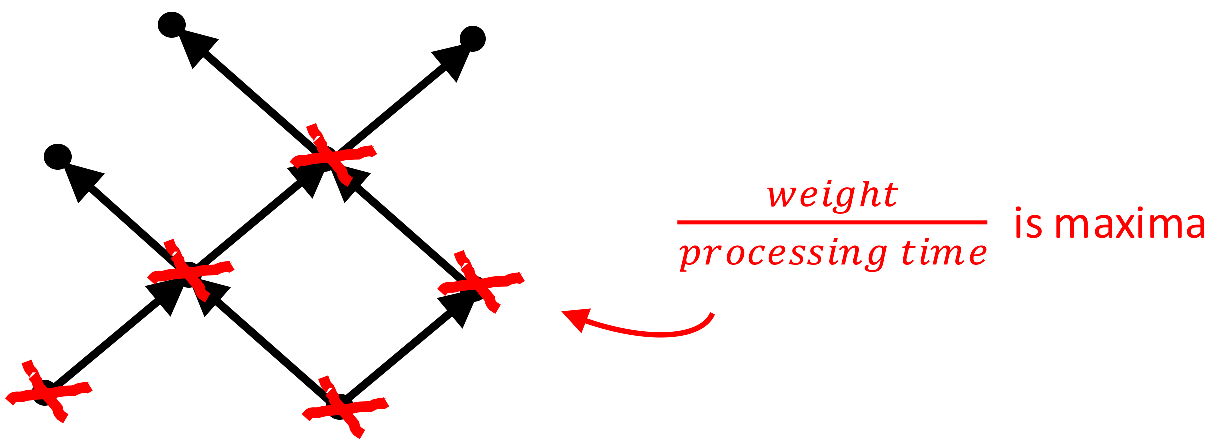



*Problem: In what order should the jobs be scheduled to minimize the sum of the weighted completion times? (*$1|prec|\sum w_i C_i$*)*

# A harder scheduling problem

*Problem: In what order should the jobs be scheduled to minimize the sum of the weighted completion times? Usually denoted* $1|prec|\sum w_i C_i$.

- Sidney (1975): an optimal schedule must begin with a closed sub-DAG $G$ of jobs that maximizes $w(G)/p(G)$. (Version of Smith's rule.)




- Recursing, this principle defines a *Sidney decomposition* of the jobs.

- If the precedence constraints are tree-like, this means the problem can be solved in polynomial time (Sidney, 1975)

# A harder scheduling problem

- Later…the problem is shown to be NP-hard problem. Many 2-approximations (schedules whose objective is within a factor 2 of that of the optimal schedule):
  - Ambühl & Mastrolilli (2009)
  - Chudak & Hochbaum (1999)
  - Chekuri & Motwani (1999)
  - Hall, Schulz, Shmoys, Wein (1997)
  - Margot, Queyranne & Wang (2003)
  - Pisaruk (2003)
  - Schulz (1996)
- In particular, Chekuri & Motwani (1999) and (independently) Margot, Queyranne & Wang (2003) showed that *any* schedule consistent with a Sidney decomposition is a 2-approximation.
- Virtually all known 2-approximations are consistent with a Sidney decomposition (Correa & Schulz, 2005)

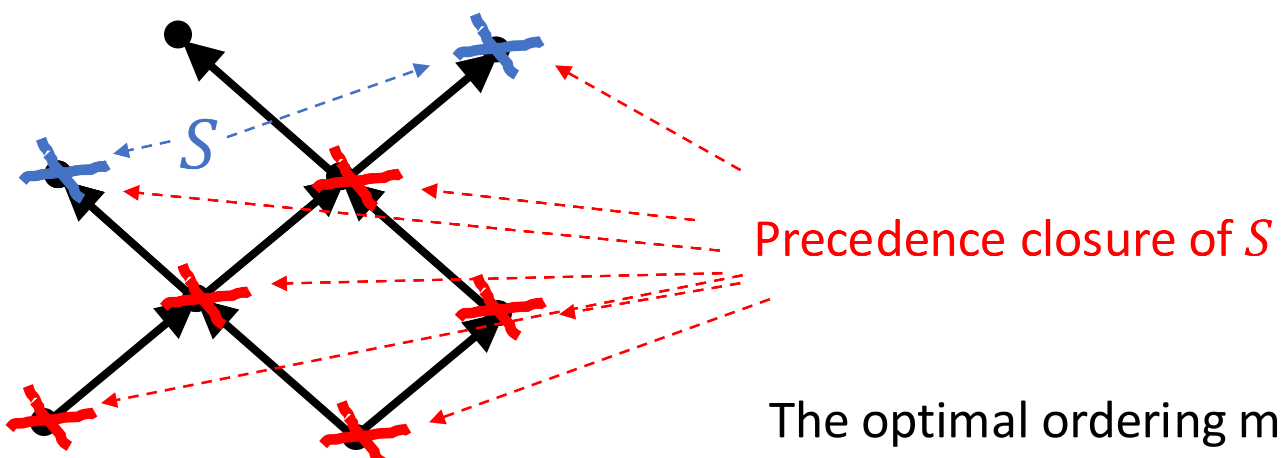
# Writing in a different form

Want to state the problem in the form: *minimize over all orderings the sum*

$$\sum_j f(S_j)\left(g(S_j) - g(S_{j-1})\right).$$

This time, take $f(S)$ = time to process all jobs in *precedence closure of* $S$,

$g(S)$ = weight of all jobs in $S$.




The optimal ordering must respect the precedence constraints.

# Writing in a different form

Want to state the problem in the form: *minimize over all orderings the sum*

$$\sum_j f(S_j)\left(g(S_j) - g(S_{j-1})\right).$$

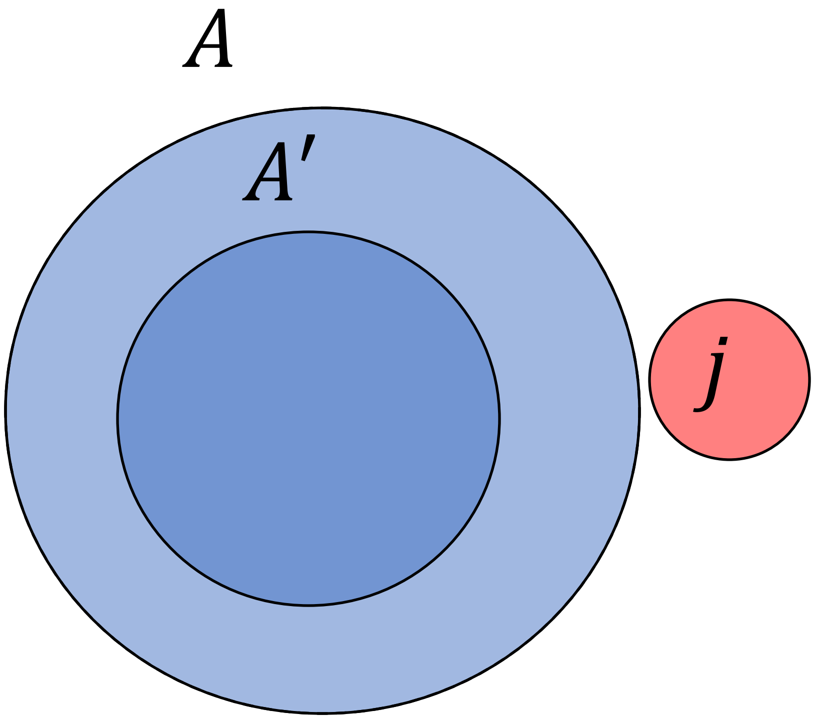
This time, take $f(S) =$ time to process all jobs in *precedence closure of* $S$,

$g(S) =$ weight of all jobs in $S$.

The function $g$ is still modular, but $f$ is *submodular* ("decreasing marginal costs"):

$$f(A \cup \{j\}) - f(A) \ \leq f(A' \cup \{j\}) - f(A') \text{ for } j \notin A \supset A'$$
$$\Leftrightarrow f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \text{ for all } A, B$$

# Submodularity




$$f(A \cup j) - f(A)$$
$$\leq$$
$$f(A' \cup j) - f(A')$$

# A new scheduling problem

- Now instead of jobs having weights, every subset $A$ of jobs has a weight $w_A$. (Weights could be equal to $0$ for most subsets.)

- We wish to minimize the weighted sum of the completion times $C_A$ of the subsets of jobs = $\sum w_A C_A$. Denote this $1|prec|\sum w_A C_A$.

- Equivalently, objects are hidden in a subset $A$ of boxes with probability $p_A$ and we wish to minimize the expected time to find all of them.

- Let $g(A) =$ sum of weights of all subsets of $A$ and let $f$ be as before. Then objective is to find an ordering to minimize

$$\sum_j f(S_j)\left(g(S_j) - g(S_{j-1})\right).$$

- Then $g$ is supermodular ("increasing marginal benefit"):
$$g(A \cup \{j\}) - g(A) \ \leq g(A' \cup \{j\}) - g(A') \text{ for } j \notin A \supset A'$$

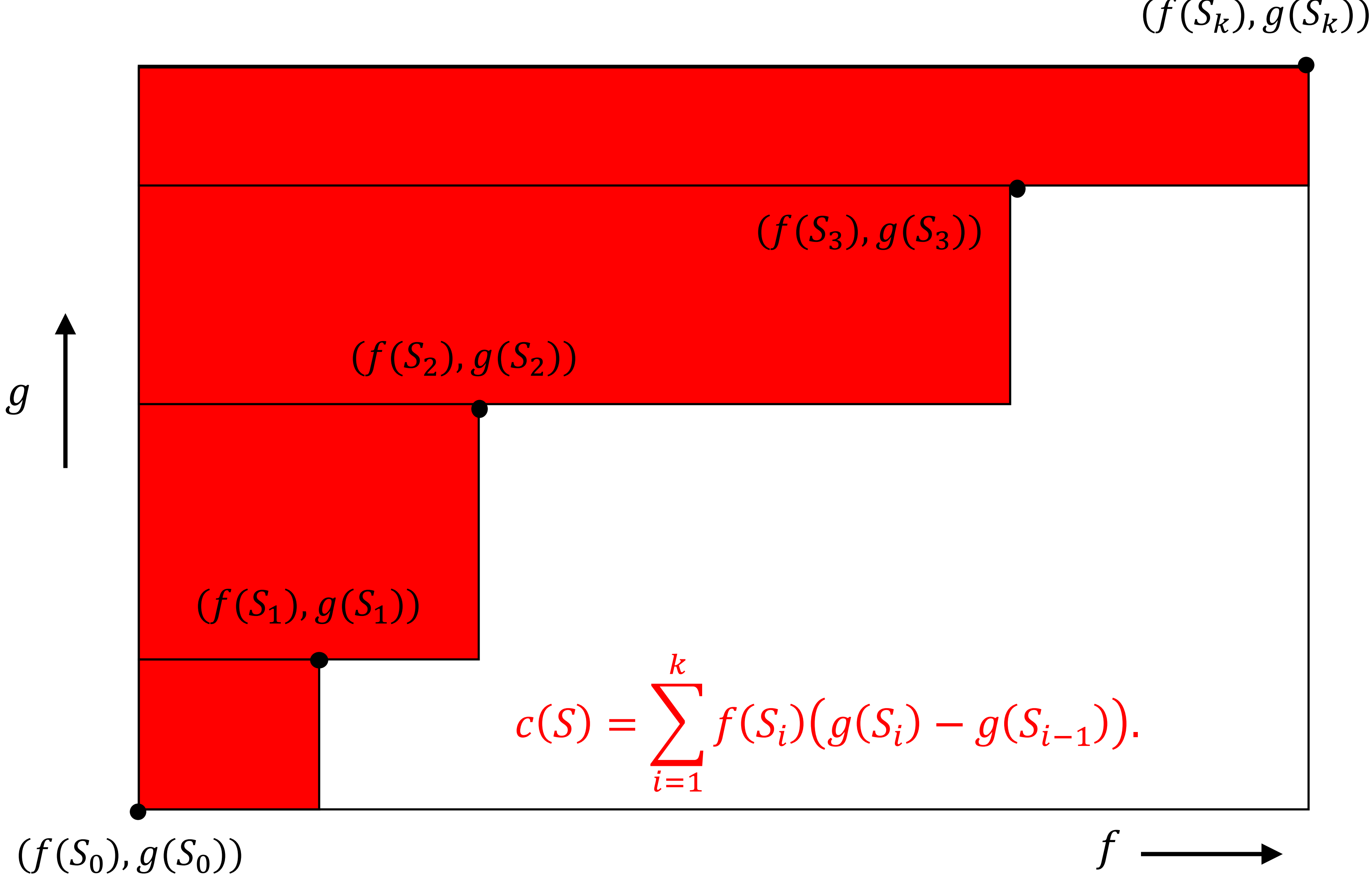
# A more general problem

MSOP with $f$ submodular, $g$ supermodular: minimize

$$\sum_{i=1}^{k} f(S_i)\big(g(S_i) - g(S_{i-1})\big).$$

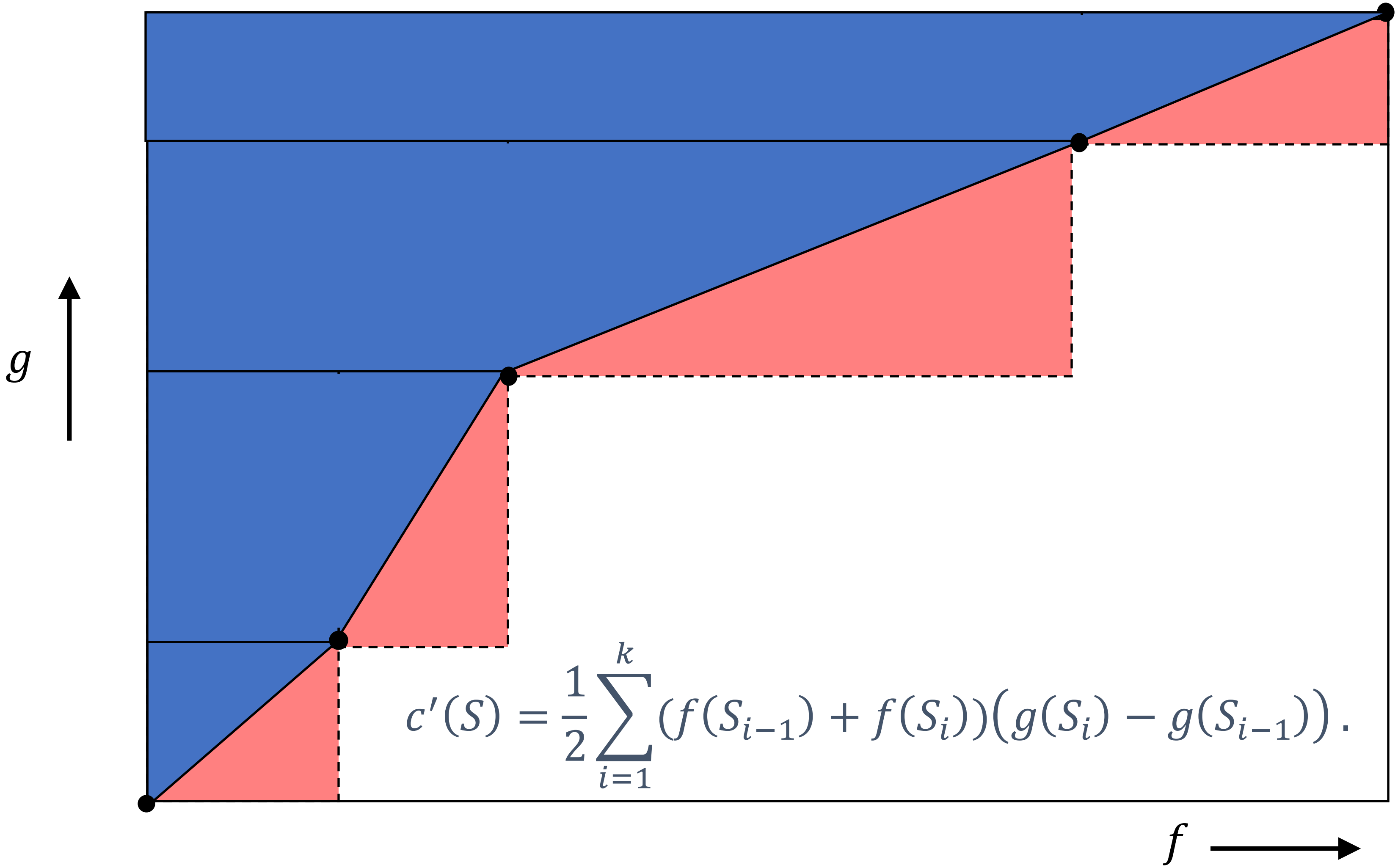
**Theorem (Pisaruk, 1992):** Any solution consistent with a "Sidney decomposition" for this problem is a 2-approximation (and a Sidney decomposition can be found in polynomial time).

# Proof idea



$(f(S_k), g(S_k))$
$(f(S_3), g(S_3))$
$(f(S_2), g(S_2))$
$g$
$(f(S_1), g(S_1))$
$c(S) = \sum_{i=1}^{k} f(S_i)\big(g(S_i) - g(S_{i-1})\big).$
$(f(S_0), g(S_0))$
$f$

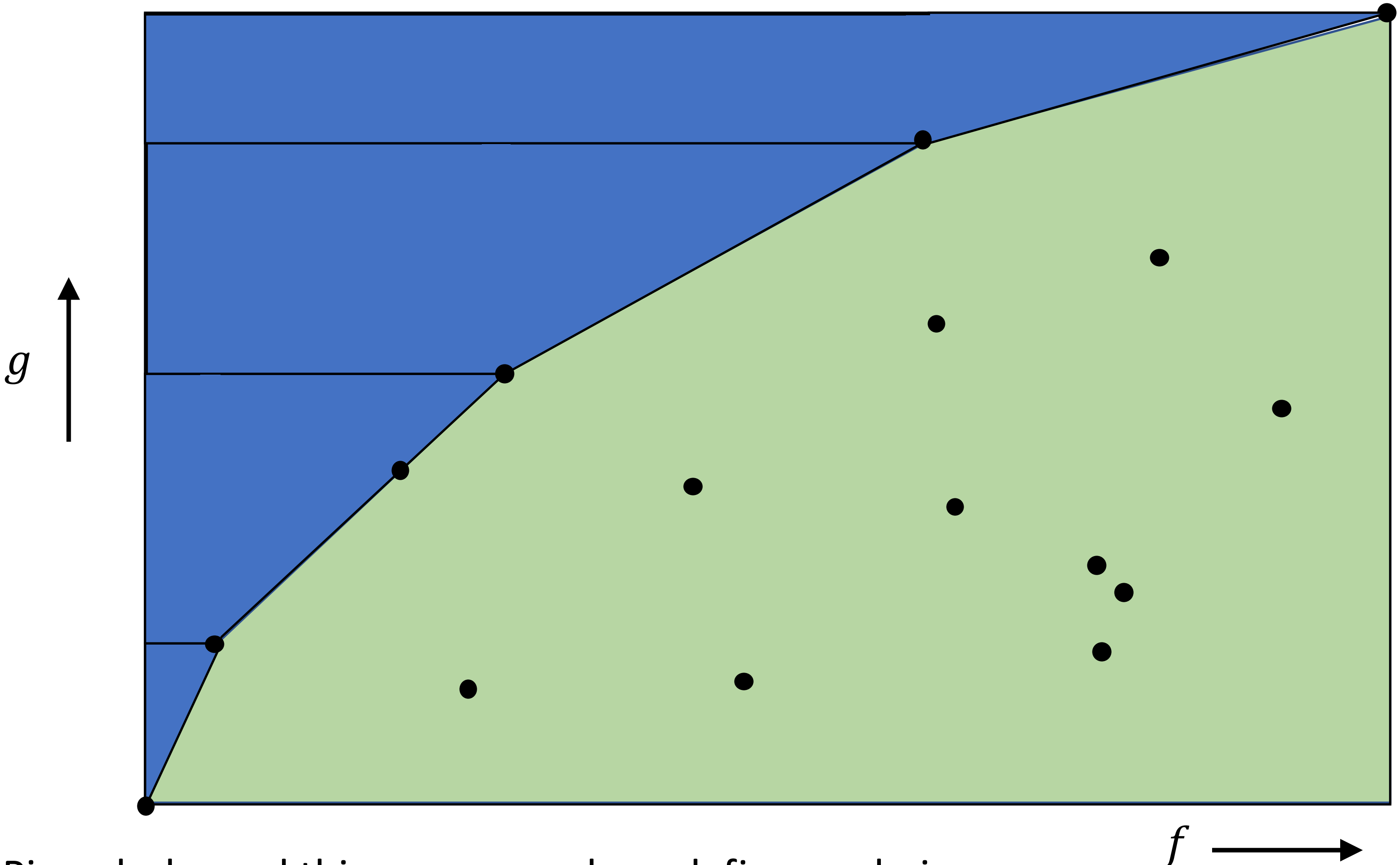
# Proof idea




$$c'(S) = \frac{1}{2}\sum_{i=1}^{k}\left(f(S_{i-1}) + f(S_i)\right)\left(g(S_i) - g(S_{i-1})\right).$$

$c(S) \leq 2c'(S)$ for any $S$

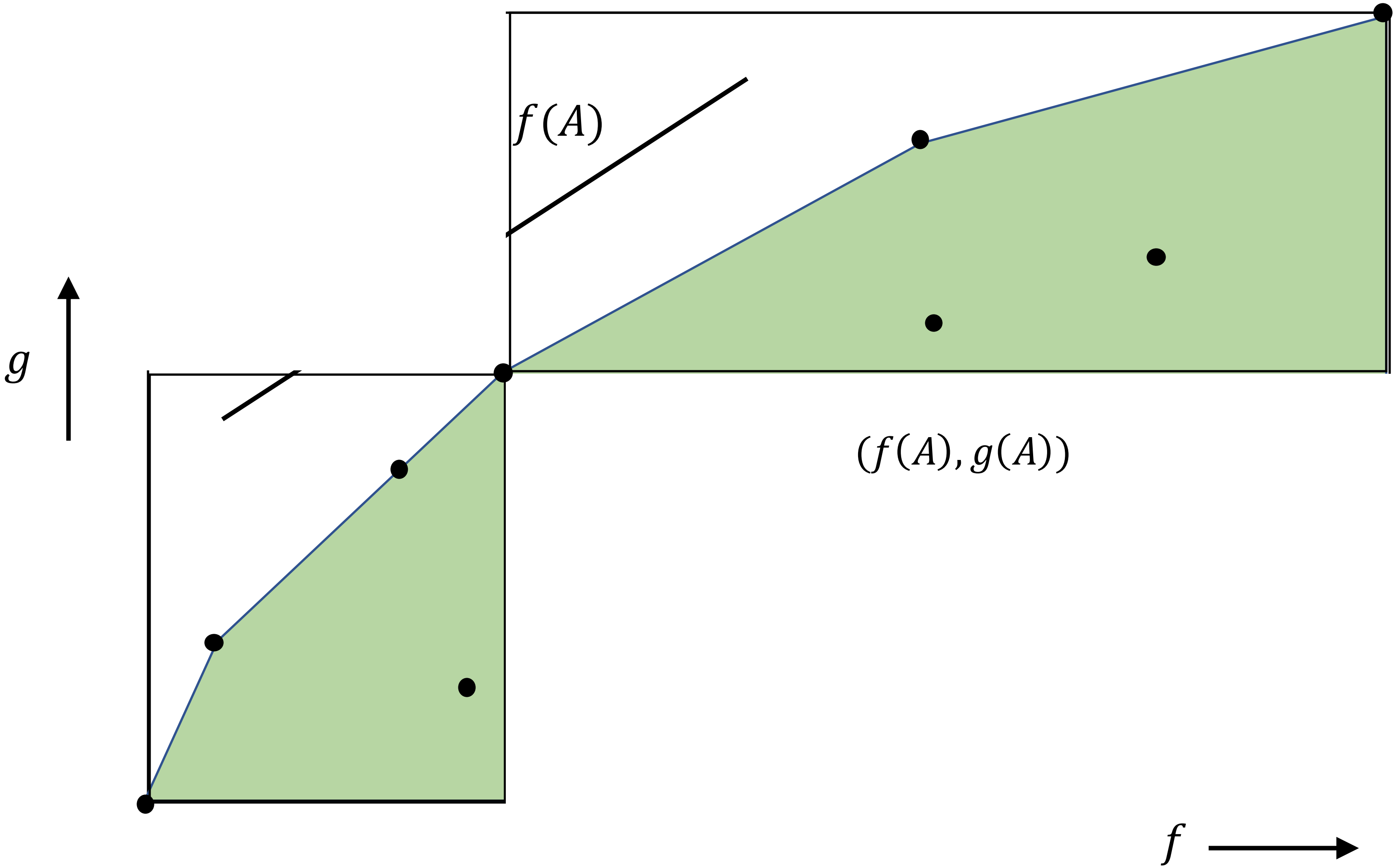
# Solution of problem of minimizing $c'$




Pisaruk showed this upper envelope defines a chain.

**Note**: the sets on the upper envelope define a Sidney decomposition.

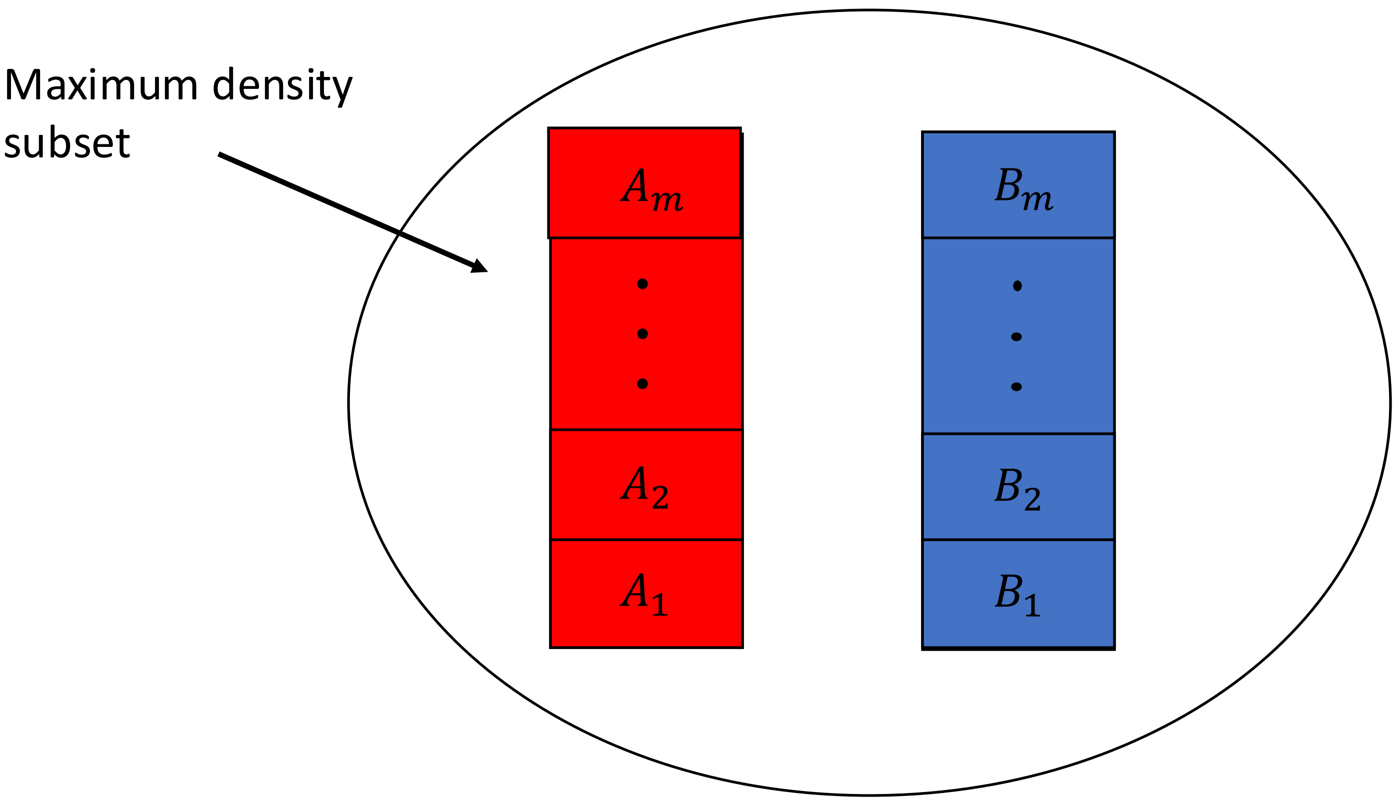
# Finding the upper envelope



# Our approach (Fokkink, L., & Végh, 2019)

**Theorem:** For $A \subset N$, let $g(A)/f(A)$ be the *density* of $A$. If $A$ has maximum density then *any* optimal search starts by searching the elements of $A$ in some order.

**Idea of proof:** Using "switching argument" to show that any optimal search is just as good as one that starts with a subset of cells with maximum density.



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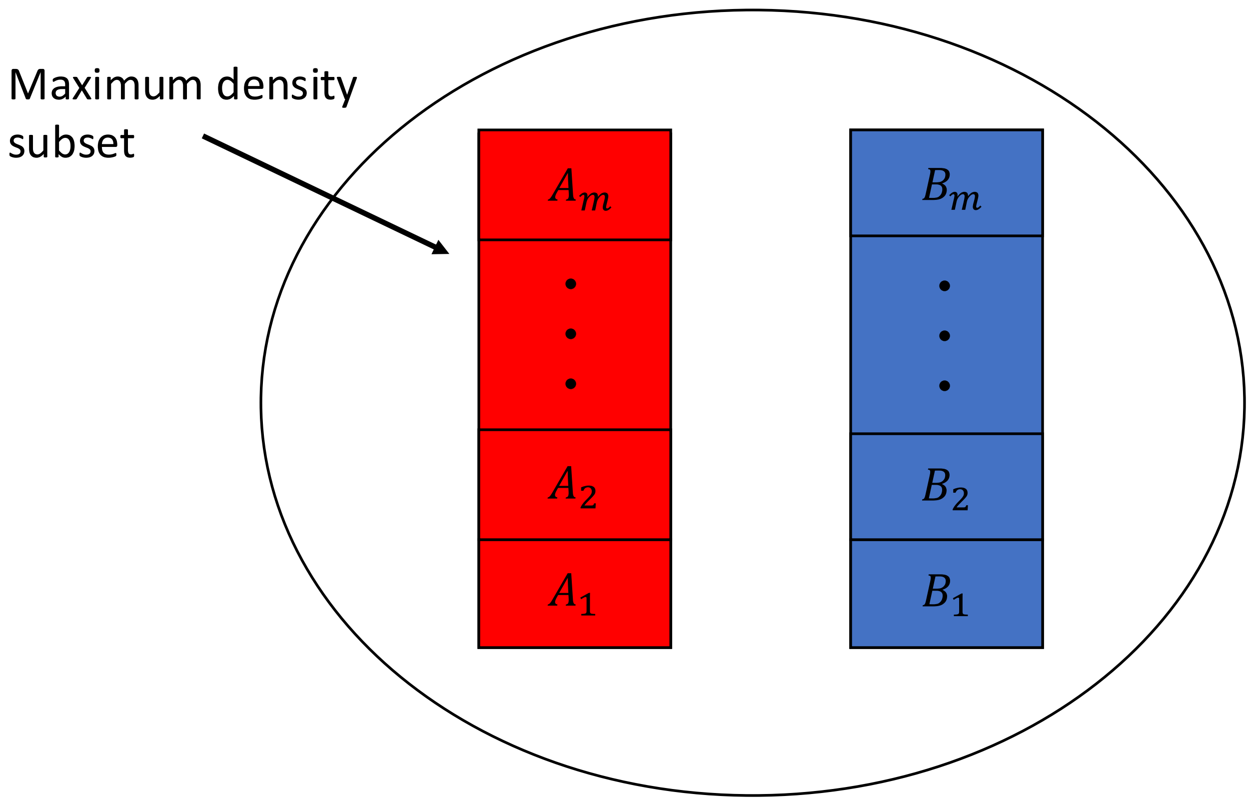
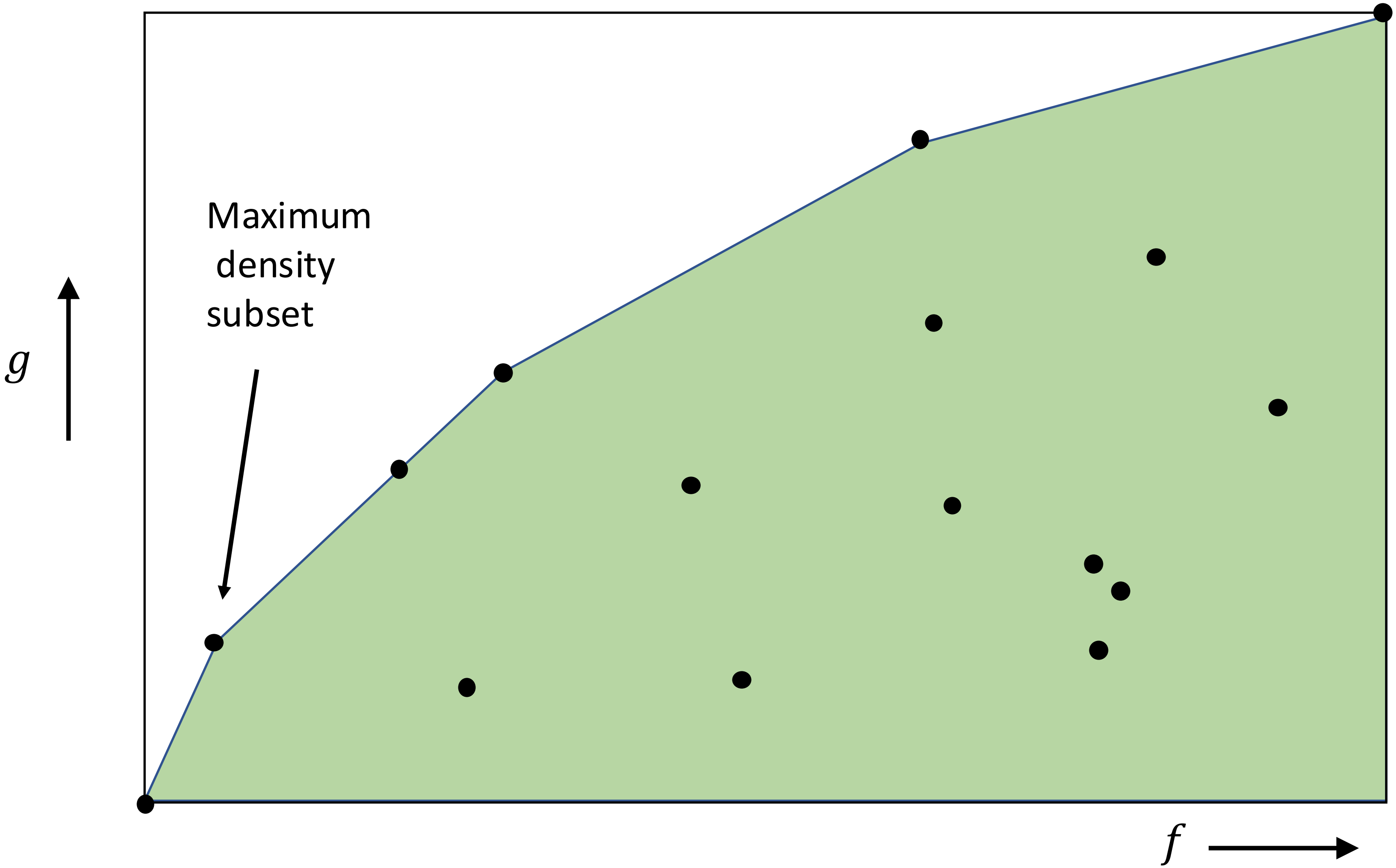



- A max. density subset can be found in polynomial time.
- Using this principle we get a Sidney decomposition.


Maximum
density
subset
$g$
$f$

# Implications for scheduling

- 2-approximation for $1|prec|\sum w_A C_A$.
- Instead of minimizing weighted sum of completions times, we could take some *function* $h$ *of* completion times, i.e. $1|prec|\sum w_A h(C_A)$.
- If $h$ is concave, the cost function $f$ is still submodular, so we get a 2-approximation.
- Note: Schulz and Verschae (2016) get a 2-approximation for the problem $1|prec|\sum w_j h(C_j)$ in a different way, using the Sidney decomposition for the problem $1|prec|\sum w_j C_j$.

# Functions of low curvature

**Definition:** The total curvature of a set function $f$ on $N$ is

$$k = 1 - \min_{i \in N} \frac{f(N) - f(N \setminus \{i\})}{f(\{i\})}$$

If $f$ is submodular, $k$ is in $[0,1]$.
Let $g^\sharp$ be the dual function of $g$, given by $g^\sharp(A) = g(N) - g(\bar{A})$. It is submodular if $g$ is supermodular.
Let $k_f$ be the total curvature of $f$ and let $k_g$ be the total curvature of $g^\sharp$.

**Theorem:** Let $\theta = (1 - k_f)(1 - k_g)$. Our algorithm for the submodular search problem has approximation ratio $\frac{2}{1+\delta}$ where

$$\delta = \min\left\{\theta, \frac{2\theta \max\{1 - k_f, 1 - k_g\}}{1 + \theta}\right\}.$$

# Implications for scheduling

- Consider the problem $1||\sum w_j h(C_j)$ with $h$ concave.
- $k_g = 0$ and

$$1 - k_f = \min_{\mathrm{i} \in N} \frac{f(N) - f(N - \{i\})}{f(\{i\})} \geq \inf_{x \in [0,1]} \frac{1 - h(1 - x)}{h(x)} = \frac{h'(1)}{h'(0)}$$

- So approximation ratio is $\frac{2}{1+\theta} \leq \frac{2}{1+\frac{h'(1)}{h'(0)}}$.

- Eg. If $h(x) = \log(1 + ax)$ , $a > 0$ then $\frac{h'(1)}{h'(0)} = \frac{1}{1+a}$ and the approximation ratio is $1 + \frac{a}{2+a}$.

- Eg. If $h(x) = 1 - \exp(-rx)$ , $r > 0$ then $\frac{h'(1)}{h'(0)} = e^{-r}$, and the approximation ratio is $\frac{2}{1+e^{-r}}$ .

# MSOP for supermodular $f$

Previous work:

- $f$ supermodular and $g$ the cardinality function: 4-approximation in Iwata, Tetali and Tripathi (2012)

- $f$ supermodular and $g$ modular: 4-approximation in Streeter and Golovin (2008)

Proofs inspired by 4-approximation for min-sum set cover in Feige, Lovász and Tetali (2004).

# Main theorem (Happach, Hellerstein & L., 2022)

**Definition:** An $\alpha$-approximate Sidney decomposition is one such that at each step, the next element of the decomposition is chosen to approximate the density problem within a factor $\alpha$.

**Theorem:** Suppose $f$ is subadditive. Then for any $\alpha$, an $\alpha$-approximate Sidney decomposition is a $4\alpha$-approximation for an optimal chain for MSOP.

$f(S \cup T) \leq f(S) + f(T)$ for disjoint $S, T$

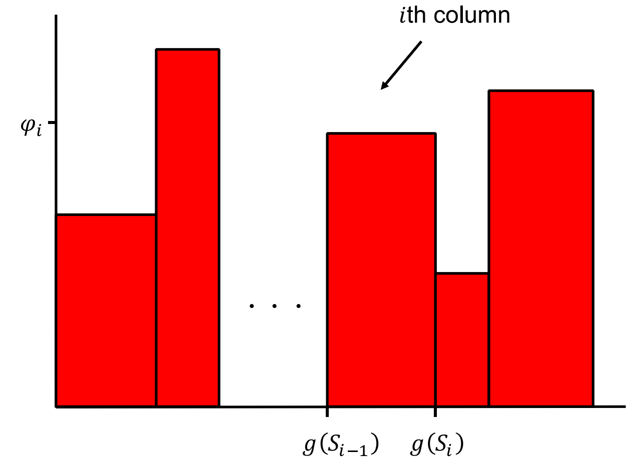
# Idea of proof of main theorem ($\alpha = 1$)

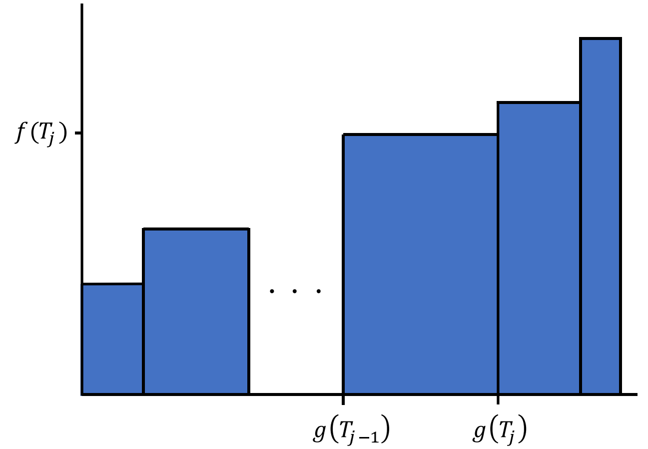
- Let $S_0, S_1, \ldots, S_k$ be a greedy chain
- Let $T_0, T_1, \ldots, T_\ell$ be an optimal chain
- Rewrite objective function for greedy chain as

$$\sum_{i=1}^{k} \varphi_i \big(g(S_i) - g(S_{i-1})\big),$$

where $\varphi_i = \big(g(V) - g(S_{i-1})\big)\frac{f(S_i)-f(S_{i-1})}{g(S_i)-g(S_{i-1})}$

- Draw a histogram for the optimal chain in the "usual way"
- Scale the red histogram by a factor of ½ in the horizontal and vertical directions and show that it fits into the blue histogram



# Idea of proof of main theorem ($\alpha = 1$)

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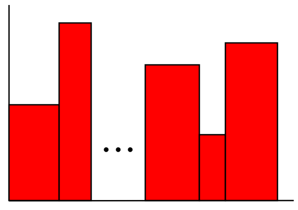

$$\sum_{i=1}^{k} \varphi_i\big(g(S_i) - g(S_{i-1})\big),$$

where $\varphi_i = \big(g(V) - g(S_{i-1})\big)\frac{f(S_i)-f(S_{i-1})}{g(S_i)-g(S_{i-1})}$

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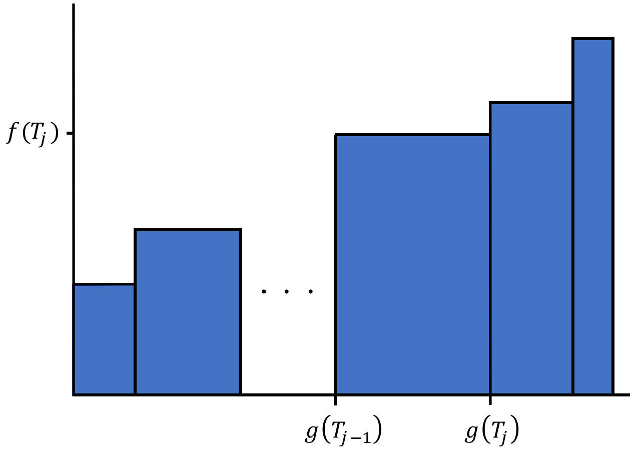


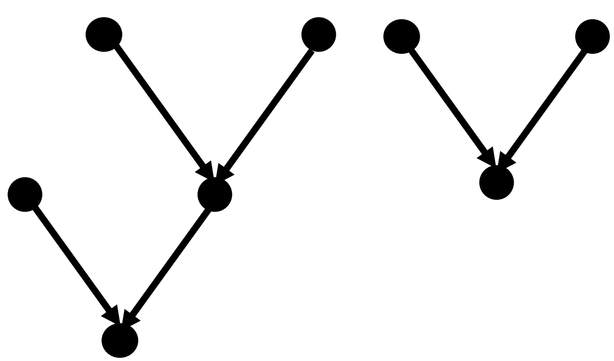
# Known applications

1. 4-approximation for $f$ supermodular and $g$ cardinality function in Iwata, Tetali and Tripathi (2012)
2. 4-approximation for $f$ supermodular and $g$ modular in Streeter and Golovin (2008)
3. 8-approximation for *expanding search* problem in Hermans, Leus and Matuschke (2021)
4. 4-approximation for bipartite OR-scheduling in Happach and Schulz (2020)

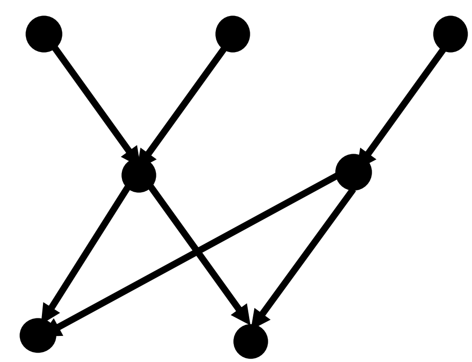
# New application: OR-scheduling

- Jobs correspond to nodes of DAG
- A job can only be completed when *at least one* of its predecessors has been completed

**Theorem:** There is a polynomial time 4-approximation algorithm for the problem of minimizing the sum of weighted completion times of a set of jobs that must be scheduled to respect some OR-precedence constraints in the form of an *inforest* (or, more generally, a *multitree*).



an *inforest*



a multitree

# Future work

- More work needed on the “density problem”
- Could a 4-approximation be found for *Generalized Min-Sum Set Cover?* Best known approximation is 4.642 (Bansal, Batra, Farhadi, Tetali, 2021)