Single-machine scheduling with an external resource

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External resources

Resources that are rented

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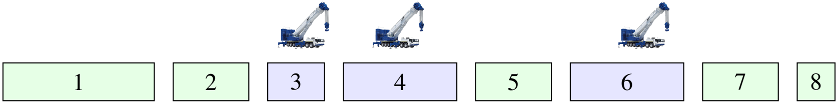
Human experts invited in projects

Problem definition



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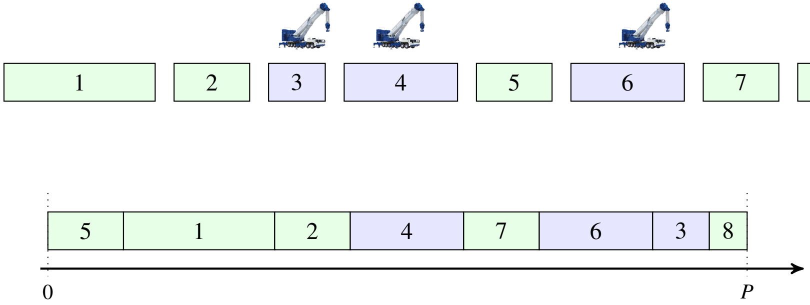

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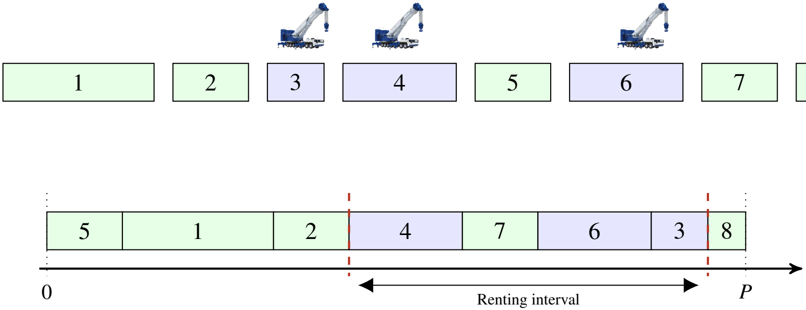

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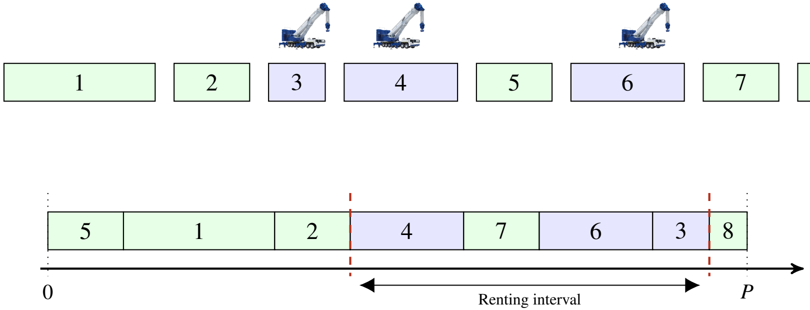

We have two type of costs

1. The scheduling cost (γ)
2. The renting cost (er)

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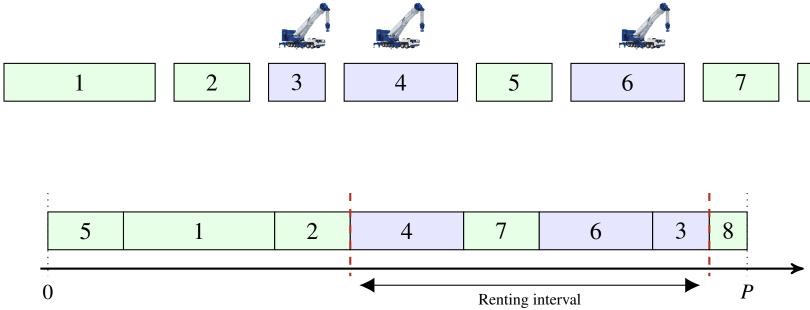

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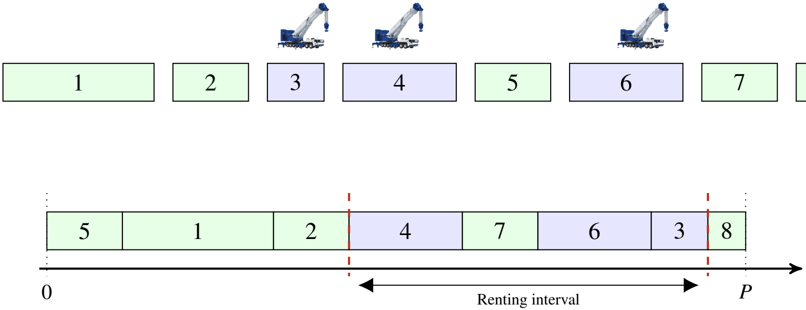

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1. The scheduling cost (γ) — $\gamma = \sum C_j, \sum w_jC_j, \max L_j, \sum w_jU_j$
2. The renting cost (er) — Linear function of length of the renting interval

Problem definition

Four variants corresponding to $1||\gamma$

Problem	To optimize	To respect
$1\|er\|\gamma$	the scheduling cost (γ)	renting interval length of at most K^r

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$1\|\gamma\|er$	renting interval length (er)	scheduling cost at most K^γ
$1\|\|(\gamma, er)$	Pareto-front of γ and er	-
$1\|\|\gamma + \lambda \cdot er$	the sum of γ and $\lambda \cdot er$	-

*: $\lambda \geq 0$ denotes the renting cost per time unit.

Problem	γ	Complexity
$1\|\|\gamma$	$\sum C_j$	$O(n \log n)$ (Smith [4])
	$\sum w_jC_j$	$O(n \log n)$ (Smith [4])
	$\max L_j$	$O(n \log n)$ (Jackson [2])
	$\sum w_jU_j$	$O(nP)$ (Lawler and Moore [3])
$1\|er\|\gamma$	$\sum C_j$	Open
	$\sum w_jC_j$	Open
	$\max L_j$	Open
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$1\|\gamma\|er$	$\sum C_j$	Open
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$1\|\|(\gamma, er)$	$\sum C_j$	Open
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	$\max L_j$	Open
	$\sum w_jU_j$	Open
$1\|\|\gamma + er$	$\sum C_j$	Open
	$\sum w_jC_j$	Open
	$\max L_j$	Open
	$\sum w_jU_j$	Open

$1|er|\sum w_jC_j$

$1|er|\sum C_j$ is NP-hard

EVEN-ODD-PARTITION: Given integers $a_1, \ldots, a_{2m}$ with $a_{k-1} < a_k$ for $k = 2, \ldots, 2m$ and with total value $2B$, is there a subset of m of these numbers with total value of B such that for each $k = 1, ..., m$ exactly one of the pair $\{a_{2k-1}, a_{2k}\}$ is in the subset?
Given such an instance of EVEN-ODD-PARTITION, we construct an instance of $1|er|\sum C_j$ with $2m + 2$ jobs as follows:

- $J = \{1, \ldots, 2m + 2\}$ and $J^r = \{2m + 1, 2m + 2\}$,
- $p_j = B^2 + a_j$ for each $j = 1, \ldots, 2m$,
- $p_{2m+1} = 0$ and $p_{2m+2} = C + D + 1$, and
- $K^r = p_{2m+2} + mB^2 + B$

where

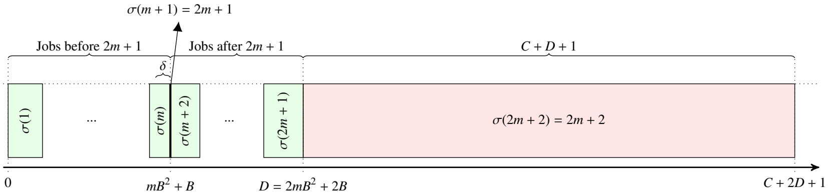
$$C = \sum_{k=1}^{m} (m + 1 - k)(p_{2k-1} + p_{2k}) + (mB^2 + B)(m + 1)$$

and

$$D = \sum_{j=1}^{2m} p_j = 2mB^2 + 2B.$$

Claim: There is a feasible schedule with total completion time of no more than $2(C + D) + 1$ if and only if the answer to the instance of EVEN-ODD-PARTITION is yes.

$1|er|\sum C_j$ is NP-hard



Claim

Job $2m+2$ is the last job in σ and job $2m+1$ is not started before $mB^2 + B$.

Claim

Exactly m jobs are scheduled between $2m+1$ and $2m+2$ in σ.

An example of 8 jobs

$p_1 = 8$	$p_2 = 4$	$p_3 = 3$	$p_4 = 6$	$p_5 = 4$	$p_6 = 6$	$p_7 = 4$	$p_8 = 2$
$w_1 = 16$	$w_2 = 8$	$w_3 = 3$	$w_4 = 24$	$w_5 = 28$	$w_6 = 12$	$w_7 = 12$	$w_8 = 20$

An example of 8 jobs

	1	2	3	4	5	6	7	8
	$p_1 = 8$	$p_2 = 4$	$p_3 = 3$	$p_4 = 6$	$p_5 = 4$	$p_6 = 6$	$p_7 = 4$	$p_8 = 2$
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$w_i/p_i =$	2	2	1	4	7	2	3	10

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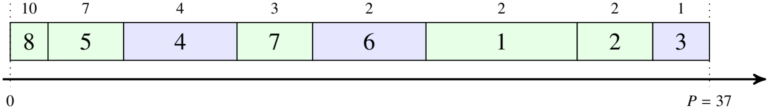
Step 1: Create a sequence according to WSPT↓



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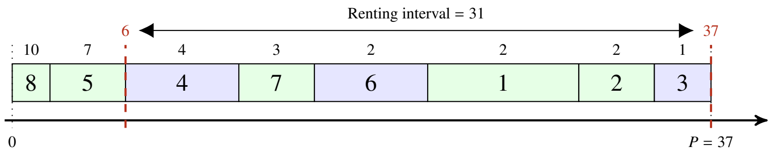

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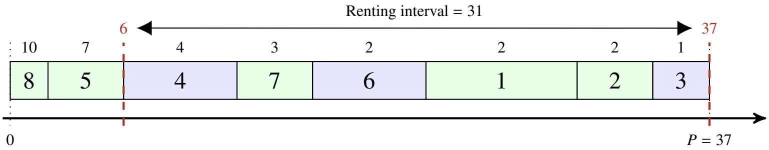

If $K^r \geq 31$, this sequence is optimal.

But, what if $K^r < 31$?

What if $K^r = 23$?

Step 2: Re-number jobs according to WSPT.

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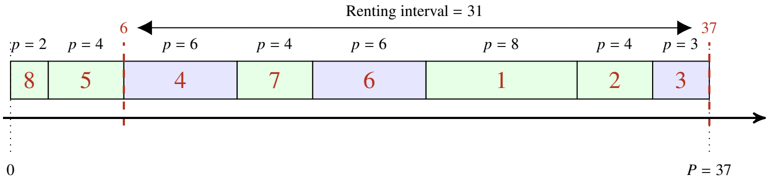

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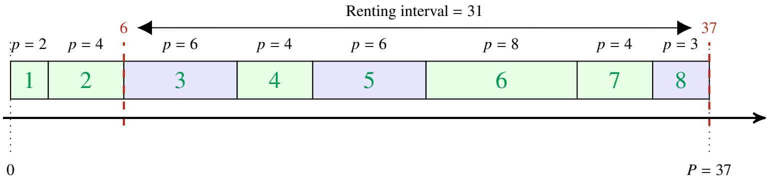

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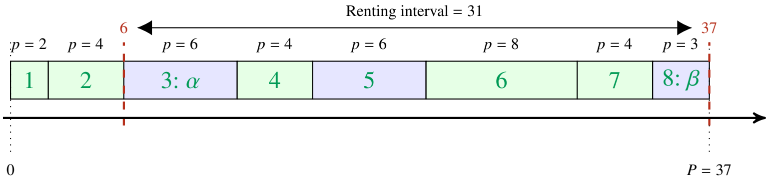

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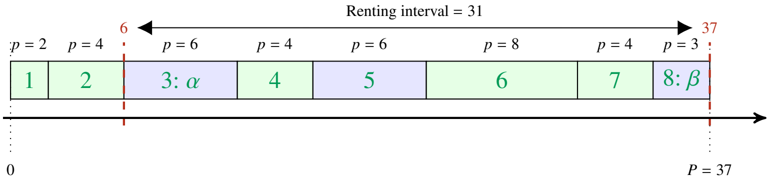

Step 3: Form set H of *ordinary jobs* between α and β.

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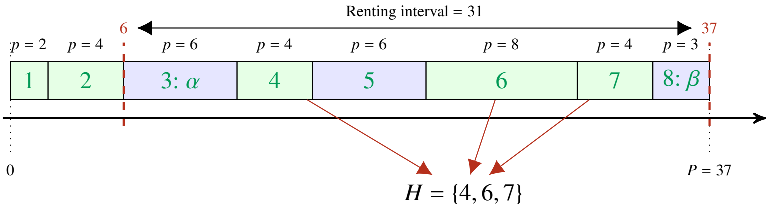

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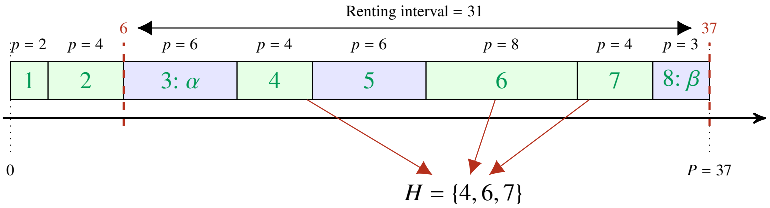

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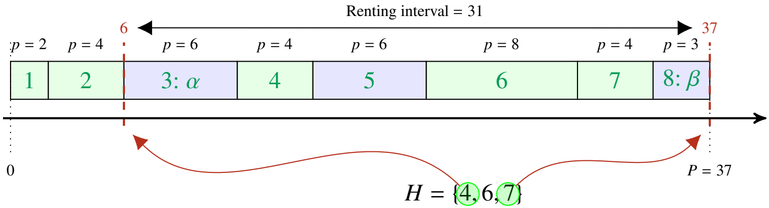

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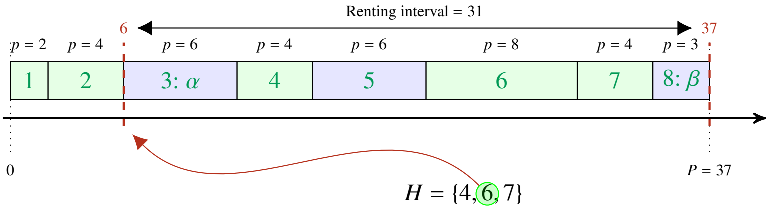

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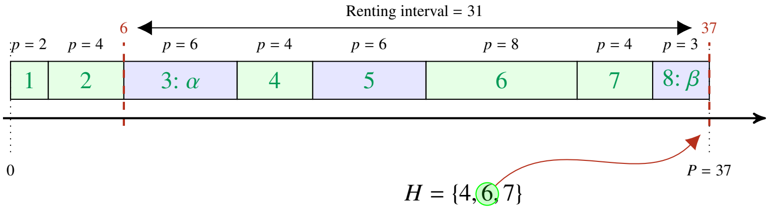

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Three different scenarios

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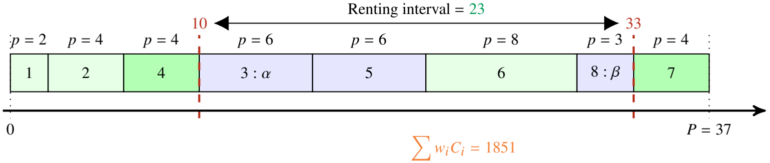

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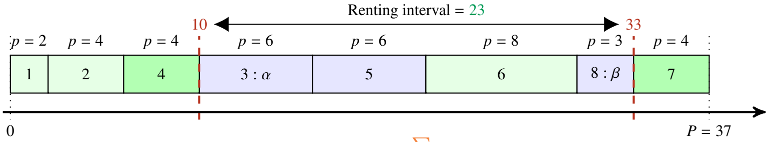
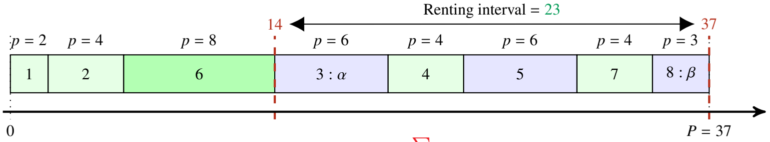


$\sum w_i C_i = 1851$

Scenario 2:



$\sum w_i C_i = 1943$

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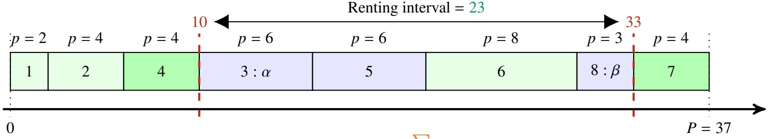

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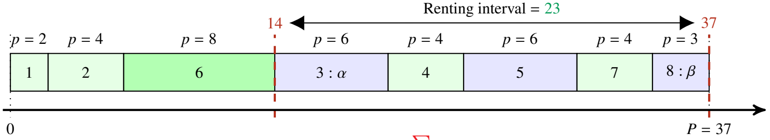

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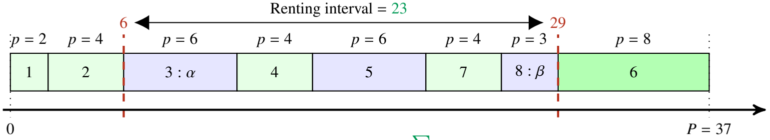
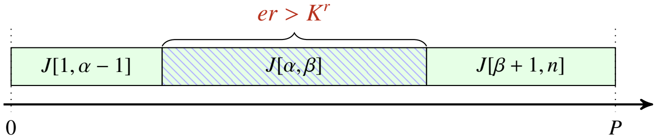


$\sum w_iC_i = 1775$

$\sigma \longrightarrow \sigma_{X,Y}$

Given sequence σ (the WSPT sequence)



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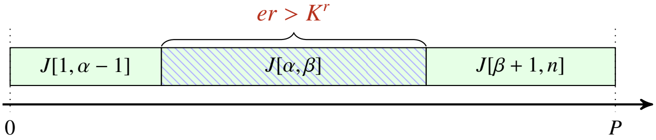

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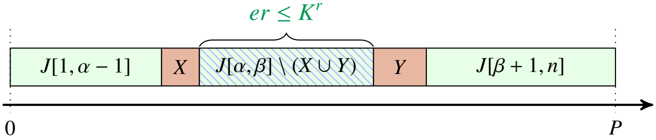

I try to find sequence $\sigma_{X,Y}$

that minimizes $\sum w_iC_i$.

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Splitting the Renting interval

Lemma

For each instance of $1|er|\sum w_jC_j$ *there exists* $X^*, Y^* \subseteq H$ *with* $\max X^* < \min Y^*$ *such that* σ_{X^*,Y^*} *is optimal.*

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For each instance of $1|er|\sum w_jC_j$ there exists $X^*, Y^* \subseteq H$ with $\max X^* < \min Y^*$ such that σ_{X^*,Y^*} is optimal.

The benefit is that it shows: there is κ^* such that

$$X^* \subseteq H \cap J[\alpha+1, \kappa^*-1]$$

and

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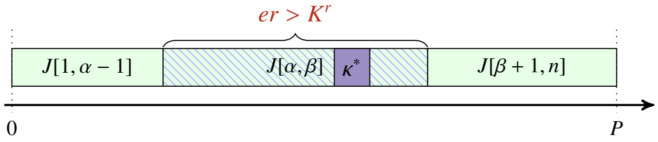

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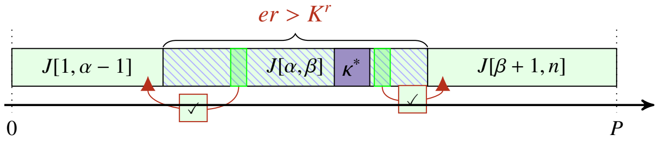

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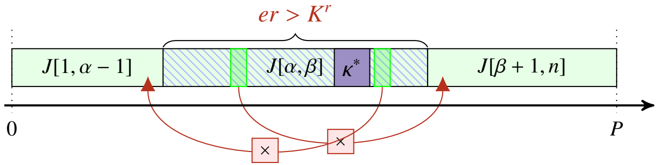

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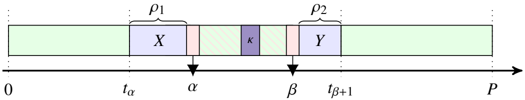
Let us construct X^* and Y^*



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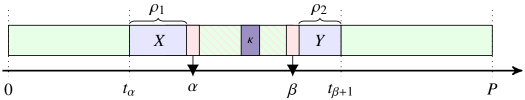
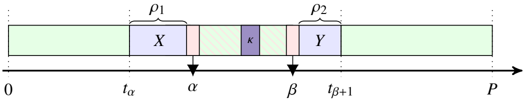


For any given trio of values κ, ρ_1, and ρ_2, we want to find the best X and Y. And then

$$X \to X_{\kappa,\rho_1} \text{ and } Y \to Y_{\kappa,\rho_2}$$

Let us construct X^* and Y^*



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X_{κ,ρ_1} and Y_{κ,ρ_2} for every tuple (κ, ρ_1, ρ_2)

$$X_{\kappa,\rho_1} \in \arg\min_{X \in \mathcal{X}_{\kappa,\rho_1}} \{f_\kappa(X)\} \quad \text{and} \quad Y_{\kappa,\rho_2} \in \arg\min_{Y \in \mathcal{Y}_{\kappa,\rho_2}} \{g_\kappa(Y)\}.$$

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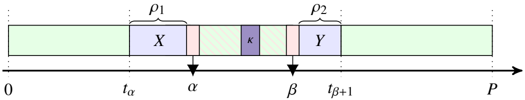

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We find the optimal triple (κ, ρ_1, ρ_2)

$$(\kappa^*, \rho_1^*, \rho_2^*) \in \arg\min_{(\kappa,\rho_1,\rho_2)|p(J[\alpha,\beta])-\rho_1-\rho_2 \leq K^r} \{f_\kappa(X_{\kappa,\rho_1}) + g_\kappa(Y_{\kappa,\rho_2})\}$$

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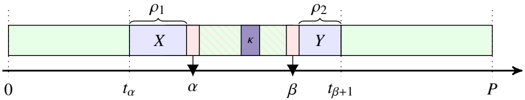

For any given trio of values κ, ρ_1, and ρ_2, we want to find the best X and Y. And then

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X_{κ,ρ_1} and Y_{κ,ρ_2} for every tuple (κ, ρ_1, ρ_2)

$$X_{\kappa,\rho_1} \in \arg\min_{X \in \mathcal{X}_{\kappa,\rho_1}} \{f_\kappa(X)\} \quad \text{and} \quad Y_{\kappa,\rho_2} \in \arg\min_{Y \in \mathcal{Y}_{\kappa,\rho_2}} \{g_\kappa(Y)\}.$$

Done in $O(nP^2)$ → next slide

We find the optimal triple (κ, ρ_1, ρ_2)

$$(\kappa^*, \rho_1^*, \rho_2^*) \in \arg\min_{(\kappa,\rho_1,\rho_2)|p(J[\alpha,\beta])-\rho_1-\rho_2 \le K^r} \{f_\kappa(X_{\kappa,\rho_1}) + g_\kappa(Y_{\kappa,\rho_2})\}$$

Done in $O(nP)$.

Dynamic programming for computing $X_{\kappa,\rho}$

Computing $X_{\kappa,\rho}$

$$\theta_{1,\rho}(\alpha-1,\varrho) = \begin{cases} 0 & \text{if } \varrho = 0 \\ \infty & \text{otherwise} \end{cases},$$

$$\theta_{1,\rho}(j,\varrho) = \min\left\{ \begin{array}{l} \begin{cases} \theta_{1,\rho}(j-1,\varrho-p_j) + w_j\cdot(t_\alpha+\varrho) & \text{if } j\in J^o \\ \infty & \text{if } j \in J^r \end{cases} \\ \theta_{1,\rho}(j-1,\varrho) + w_j\cdot(p(J[1,j]) + \rho - \varrho) \end{array} \right\}.$$

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So for all (κ,ρ), the algorithm runs in $O(n^2P^2)$?

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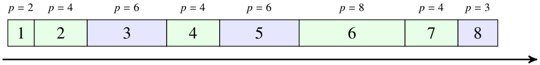

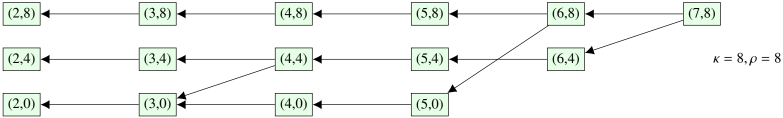

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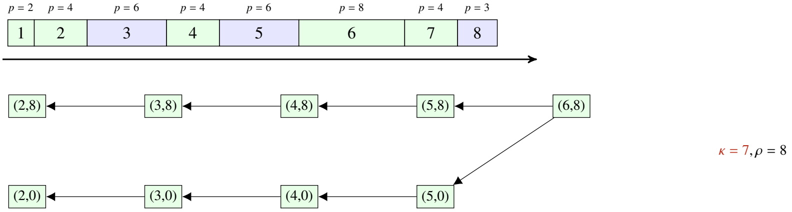

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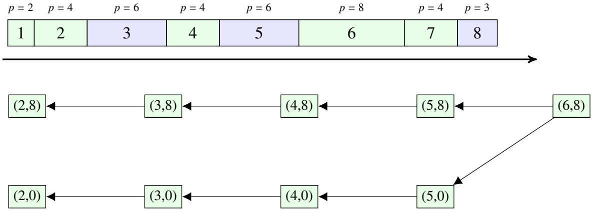

$\kappa = 7, \rho = 8$

Final Result for $1|er|\sum w_jC_j$ and $1|er|\sum C_j$

Lemma

$1|er|\sum w_jC_j$ can be solved in $O(nP^2)$-time.

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$1|er|\sum w_jC_j$ *can be solved in* $O(nP^2)$*-time.*

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$1|er|\sum C_j$ *can be solved in* $O(n^2P)$*-time.*

$1||\sum w_jC_j + \lambda \cdot er$

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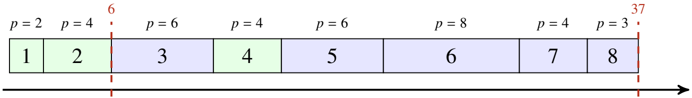

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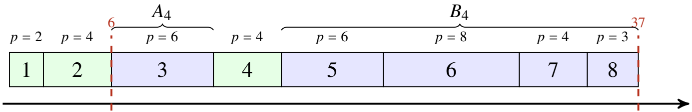

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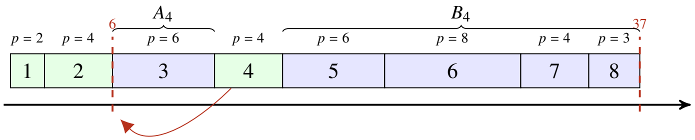

$$\underbrace{w(A_4)p_4 - \lambda \cdot p_4}_{\text{moving } A_4 \text{ forward}} - \underbrace{w_4 P(A_4)}_{\text{moving 4 backward}} < 0$$

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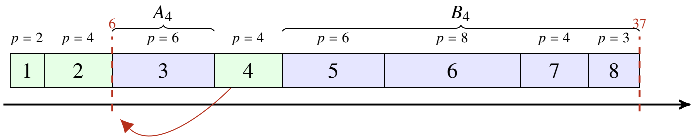

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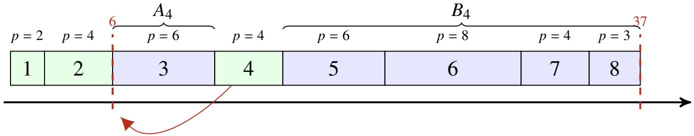

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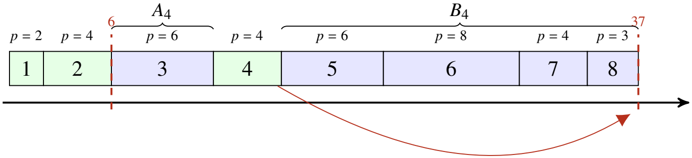

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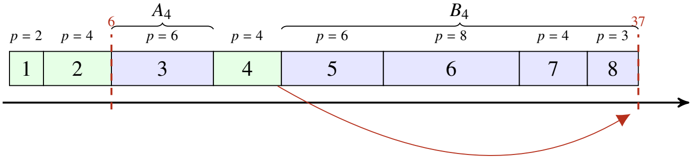

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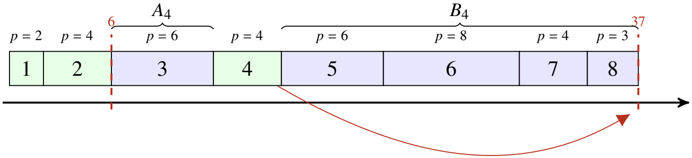

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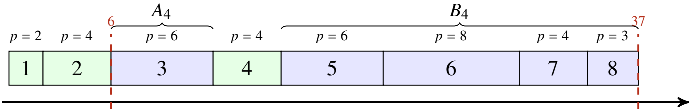
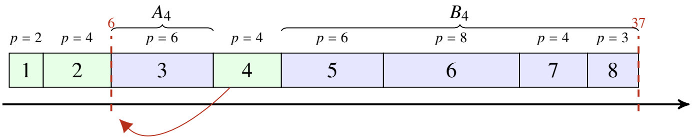
$1||\sum w_jC_j + \lambda \cdot er$ for job 4



$1||\sum w_jC_j + \lambda \cdot er$ for job 4



So, if

$$\frac{w_4}{p_4} > \frac{w(A_4) - \lambda}{p(A_4)} \quad \text{and} \quad \frac{w_4}{p_4} \geq \frac{w(J^r)}{p(J^r)},$$

then 4 is moved before the Renting interval ($\rightarrow X$).

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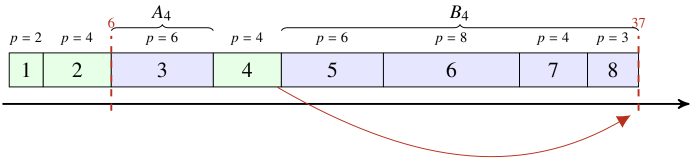

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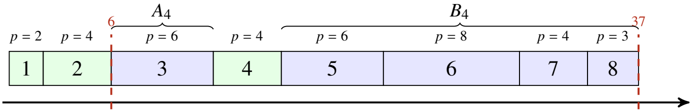

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then 4 is moved after the Renting interval ($\rightarrow Y$).

Otherwise 4 stays inside the Renting interval.

$1||\sum w_jC_j + \lambda \cdot er$

$$X_\lambda := \left\{ j \in H : \frac{w_j}{p_j} > \frac{w(A_j) - \lambda}{p(A_j)} \text{ and } \frac{w_j}{p_j} \geq \frac{w(J^r)}{p(J^r)} \right\},$$

$$Y_\lambda := \left\{ j \in H : \frac{w_j}{p_j} < \frac{w(B_j) + \lambda}{p(B_j)} \text{ and } \frac{w_j}{p_j} < \frac{w(J^r)}{p(J^r)} \right\}.$$

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Lemma

$\sigma_{X_\lambda,Y_\lambda}$ is an optimal sequence for $1||\sum w_jC_j + er$ with unit rental cost λ.

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Theorem

$1||\sum w_jC_j + \lambda \cdot er$ can be solved in time $O(n \log n)$.

$1|er|\max L_j$

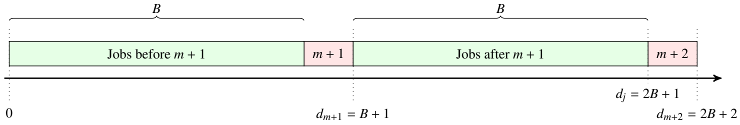
$1|er|\max L_j$ is NP-hard

PARTITION: Given integer numbers $a_1, \ldots, a_m$, is there a subset of $\{a_1, \ldots, a_m\}$ with total value of $B = \frac{1}{2}\sum_{i=1}^{m} a_i$?

Given an instance of PARTITION, we construct an instance of $1|er|\max L_j$ with $m+2$ jobs as follows:

- $J = \{1, \ldots, m+2\}$ and $J^r = \{m+1, m+2\}$,
- $p_j = a_j$ and $d_j = 2B + 1$ for each $j = 1, \ldots, m$,
- $p_{m+1} = 1$, $d_{m+1} = B + 1$, $p_{m+2} = 1$ and $d_{m+2} = 2B + 2$, and
- $K^r = B + 2$.

$1|er|\max L_j$ is NP-hard

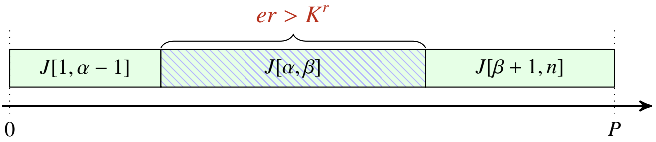


Claim

There is a feasible schedule with maximum lateness of at most zero if and only if the answer to the instance of PARTITION *is yes.*

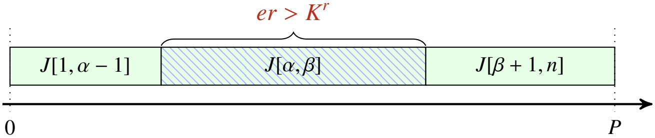
$\sigma \rightarrow \sigma_{X,Y}$

Given sequence σ (the EDD sequence)



**Remember: Jobs are re-numbered according to EDD.
**Note: $S = X \cup Y \subseteq H$.

$\sigma \to \sigma_{X,Y}$



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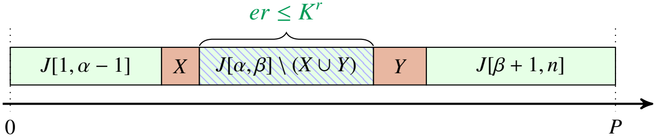

I try to find sequence $\sigma_{X,Y}$

that minimizes $\max L_j$.

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Splitting the Renting interval

Lemma

For each instance of $1|er|\max L_j$ *there exists* $X^*, Y^* \subseteq H$ *with* $\max X^* < \min Y^*$ *such that* σ_{X^*,Y^*} *is optimal.*

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The benefit is that it shows: there is κ^* such that

$$X^* \subseteq H \cap J[\alpha + 1, \kappa^* - 1]$$

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$$Y^* \subseteq H \cap J[\kappa^*, \beta - 1].$$

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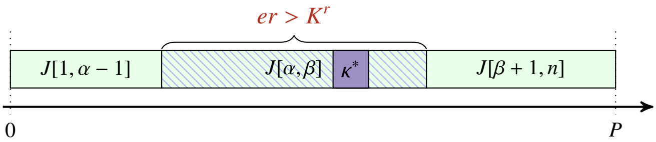

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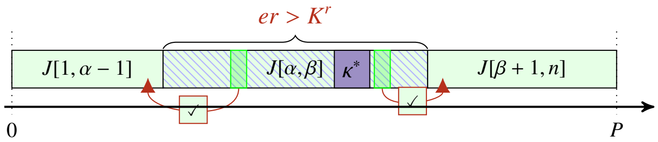

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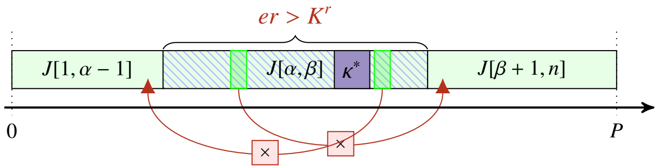

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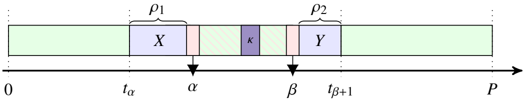
Let us construct X^* and Y^*



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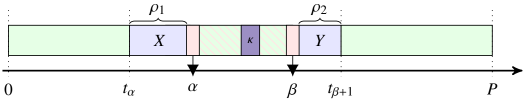

For any given trio of values κ, ρ_1, and ρ_2, we want to find the best X and Y. And then

$$X \to X_{\kappa,\rho_1} \text{ and } Y \to Y_{\kappa,\rho_2}$$

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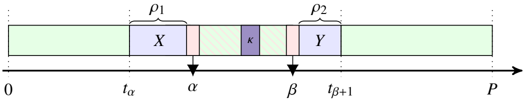

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X_{κ,ρ_1} and Y_{κ,ρ_2} for every tuple (κ, ρ_1, ρ_2)

$$X_{\kappa,\rho_1} \in \arg\min_{X \in \mathcal{X}_{\kappa,\rho_1}} \{f'_\kappa(X)\} \quad \text{and} \quad Y_{\kappa,\rho_2} \in \arg\min_{Y \in \mathcal{Y}_{\kappa,\rho_2}} \{g'_\kappa(Y)\}.$$

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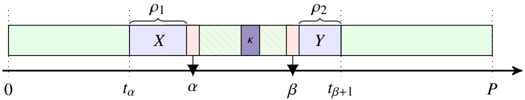

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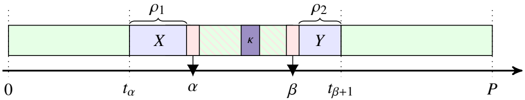
We find the optimal triple (κ, ρ_1, ρ_2)

$$(\kappa^*, \rho_1^*, \rho_2^*) \in \arg\min_{(\kappa,\rho_1,\rho_2) | p(J[\alpha,\beta]) - \rho_1 - \rho_2 \le K^r} \{\max\{f'_\kappa(X_{\kappa,\rho_1}), g'_\kappa(Y_{\kappa,\rho_2})\}\}$$

Let us construct X^* and Y^*



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Done in $O(nP)$ → next slide

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Done in $O(nP)$.

Dynamic programming for computing $X_{\kappa,\rho}$

Accelerated jobs do not have maximum lateness

Jobs in X do not imply maximum lateness since $d_\alpha \leq d_j$ and $C_\alpha \geq C_j$ for each $j \in X$.

Computing $X_{\kappa,\rho}$ (Similarly for $Y_{\kappa,\rho}$)

$$\theta_3(\alpha+1,\rho) = \begin{cases} p(J[1,\alpha]) - d_\alpha & \text{if } \rho = 0 \\ \infty & \text{otherwise} \end{cases},$$

$$\theta_3(\kappa+1,\rho) = \min \left\{ \begin{array}{l} \max \left\{ \begin{array}{l} \theta_3(\kappa,\rho), \\ p(J[1,\kappa]) - d_\kappa \end{array} \right\}, \\ \begin{cases} \theta_3(\kappa,\rho-p_\kappa) + p_\kappa & \text{if } \kappa \in J^o \\ \infty & \text{if } \kappa \in J^r \end{cases} \end{array} \right\}.$$

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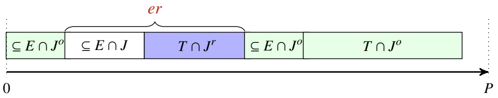
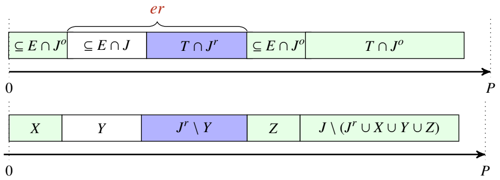
$1|er|\sum w_jU_j$

Structure of optimal schedules



Structure of optimal schedules



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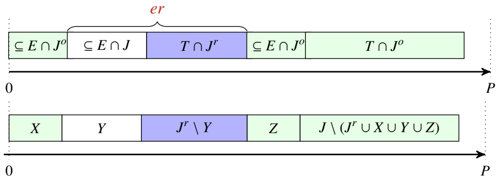
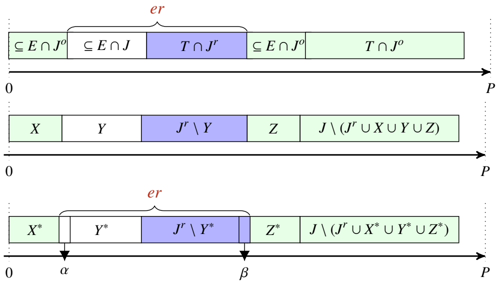


Lemma

For each instance of $1|er|\sum w_jU_j$ *there exist disjoint sets* $X^*, Y^*, Z^* \subseteq J$ *such that the sequence* σ_{X^*,Y^*,Z^*} *is optimal and, moreover,*

1. *all jobs in* $X^* \cup Y^* \cup Z^*$ *are non-tardy in* σ_{X^*,Y^*,Z^*}*,*
2. $(X^* \cup Z^*) \cap J^r = \emptyset$*, and*
3. $\max(X^* \cup (Y^* \cap J^o)) < \kappa \leq \min Z^*$.

Structure of optimal schedules



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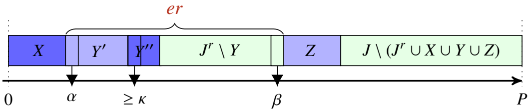
Let us construct X^*, Y^*, and Z^*



$\mathcal{X}_{\kappa,\rho_1,\rho_2}$, $\mathcal{Y}''_{\kappa,\rho_1}$, and $\mathcal{Z}_{\kappa,\rho_2}$ for every tuple (κ, ρ_1, ρ_2)

$$\mathcal{X}_{\kappa,\rho_1,\rho_2} = \left\{ (X, Y') \;\middle|\; \begin{array}{l} X, Y' \subseteq J[1, \kappa - 1],\ X \cap Y' = \emptyset = X \cap J^r, \\ \sum_{j' \in X \cup Y' : j' \leq j} p_{j'} \leq d_j, \forall j \in X \cup Y', \\ p(Y' \cap J^o) \leq K^r - p(J^r), \\ p(X \cup Y') = \rho_1,\ \rho_1 + p(J^r \setminus Y') = \rho_2 \end{array} \right\}$$

$$\mathcal{Y}''_{\kappa,\rho_1} = \left\{ Y'' \;\middle|\; Y'' \subseteq J^r[\kappa, n],\ \rho_1 + \sum_{j' \in Y'' : j' \leq j} p_{j'} \leq d_j, \forall j \in Y'' \right\}$$

$$\mathcal{Z}_{\kappa,\rho_2} = \left\{ Z \;\middle|\; Z \subseteq J^o[\kappa, n],\ \rho_2 + \sum_{j' \in Z : j' \leq j} p_{j'} \leq d_j, \forall j \in Z \right\}$$

Let us construct X^*, Y^*, and Z^*

Optimal job sets for given triple $(\kappa, \rho_1, \rho_2) \in \Xi$

$$(X_{\kappa,\rho_1,\rho_2}, Y'_{\kappa,\rho_1,\rho_2}) \in \underset{(X,Y')\in\mathcal{X}_{\kappa,\rho_1,\rho_2}}{\arg\max}\; w(X) + w(Y'),$$

$$Y''_{\kappa,\rho_1} \in \underset{Y''\in\mathcal{Y}''_{\kappa,\rho_1}}{\arg\max}\, w(Y''), \text{ and}$$

$$Z_{\kappa,\rho_2} \in \underset{Z\in\mathcal{Z}_{\kappa,\rho_2}}{\arg\max}\, w(Z)$$

Let us construct X^*, Y^*, and Z^*

Optimal job sets for given triple $(\kappa, \rho_1, \rho_2) \in \Xi$

$$(X_{\kappa,\rho_1,\rho_2}, Y'_{\kappa,\rho_1,\rho_2}) \in \operatorname*{arg\,max}_{(X,Y') \in \mathcal{X}_{\kappa,\rho_1,\rho_2}} w(X) + w(Y'),$$

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We find the optimal triple (κ, ρ_1, ρ_2)

$$(\kappa^*, \rho_1^*, \rho_2^*) \in \operatorname*{arg\,max}_{(\kappa,\rho_1,\rho_2) \in \Xi} \{w(X_{\kappa,\rho_1,\rho_2}) + w(Y'_{\kappa,\rho_1,\rho_2}) + w(Y''_{\kappa,\rho_1}) + w(Z_{\kappa,\rho_2})\}$$

Let us construct X^*, Y^*, and Z^*

Optimal job sets for given triple $(\kappa, \rho_1, \rho_2) \in \Xi$

$$
\begin{aligned}
(X_{\kappa,\rho_1,\rho_2}, Y'_{\kappa,\rho_1,\rho_2}) &\in \underset{(X,Y')\in\mathcal{X}_{\kappa,\rho_1,\rho_2}}{\arg\max}\ w(X) + w(Y'), \\
Y''_{\kappa,\rho_1} &\in \underset{Y''\in\mathcal{Y}''_{\kappa,\rho_1}}{\arg\max}\ w(Y''), \text{ and} \\
Z_{\kappa,\rho_2} &\in \underset{Z\in\mathcal{Z}_{\kappa,\rho_2}}{\arg\max}\ w(Z)
\end{aligned}
$$

We find the optimal triple (κ, ρ_1, ρ_2)

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(\kappa^*, \rho_1^*, \rho_2^*) \in \underset{(\kappa,\rho_1,\rho_2)\in\Xi}{\arg\max} \{w(X_{\kappa,\rho_1,\rho_2}) + w(Y'_{\kappa,\rho_1,\rho_2}) + w(Y''_{\kappa,\rho_1}) + w(Z_{\kappa,\rho_2})\}
$$

Done in $O(nP^4)$.

Results

Problem	γ	Complexity
$1\|\|\gamma$	$\sum C_j$	$O(n \log n)$ (Smith [4])
	$\sum w_jC_j$	$O(n \log n)$ (Smith [4])
	$\max L_j$	$O(n \log n)$ (Jackson [2])
	$\sum w_jU_j$	$O(nP)$ (Lawler and Moore [3])
$1\|er\|\gamma$	$\sum C_j$	$O(n^2 P)$
	$\sum w_jC_j$	$O(nP\max(P, W))$
	$\max L_j$	$O(nP)$
	$\sum w_jU_j$	$O(nP^4)$
$1\|\gamma\|er$	$\sum C_j$	$O(n^2 P)$
	$\sum w_jC_j$	$O(nP\max(P, W))$
	$\max L_j$	$O(nP)$
	$\sum w_jU_j$	$O(nP^4 \log(P))$
$1\|\|(\gamma, er)$	$\sum C_j$	$O(nP^2)$
	$\sum w_jC_j$	$O(nP^2)$
	$\max L_j$	$O(nP^2)$
	$\sum w_jU_j$	$O(nP^5)$
$1\|\|\gamma + er$	$\sum C_j$	$O(n \log n)$
	$\sum w_jC_j$	$O(n \log n)$
	$\max L_j$	$O(nP^2)$
	$\sum w_jU_j$	$O(nP^5)$

Future Research

- Multiple machines
- Multiple resources
- Multiple renting intervals
- Specific applications...?

References

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[3] Lawler, E. and Moore, J. M. (1969). A functional equation and its application to resource allocation and scheduling problem. *Management Science*, 16:77–84.

[4] Smith, W. E. (1956). Various optimizers for single-stage production. *Naval Research Logistics Quarterly*, 3:59–66.

Single-machine scheduling with an external resource

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