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Fairness in Repetitive Scheduling

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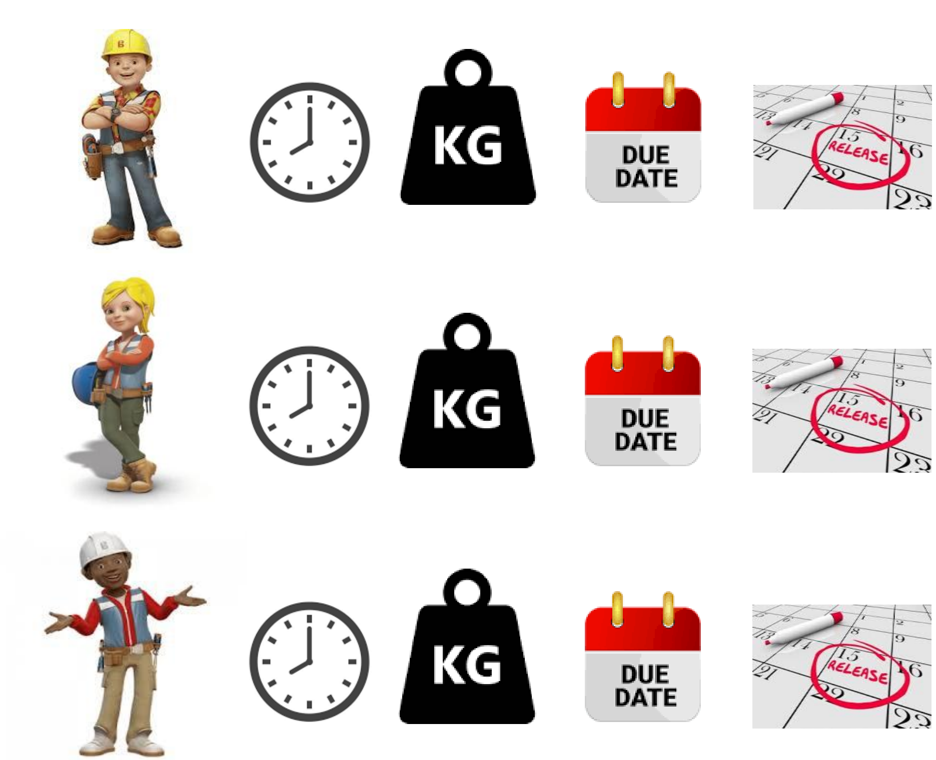
schedulingseminar.com

Classical Scheduling

- Most single machine scheduling problems look like this:
  - There are n jobs (or clients):



- Each job j has:
  - a processing time p_j,
  - a weight w_j (which may equal 1),
  - a due date d_j (sometimes),
  - a release time r_j (sometimes),
  - etc…

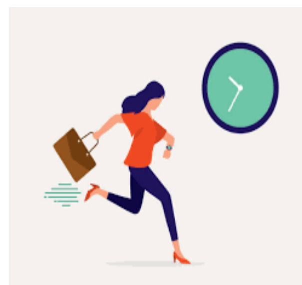
Classical Scheduling

- Most single machine scheduling problems look like this:
  - A schedule (when there are no release times) is simply a permutation of the jobs, specifying the order of processing:

- The completion time of job j is such a schedule is $C_j = \sum_{\substack{i \text{ is not after } j \\ \text{in the schedule}}} p_i$

Classical Scheduling

- The clients may have different objectives:
  - they may want to -



- minimize completion time,

  $$C_j = \sum_{\pi(i) \leq \pi(j)} p_i$$

- minimize lateness,

  $$L_j = C_j - d_j$$

- not be tardy,

  $$U_j = \begin{cases} 1 : & C_j > d_j \\ 0 : & C_j \leq d_j \end{cases}$$

- etc…

Classical Scheduling

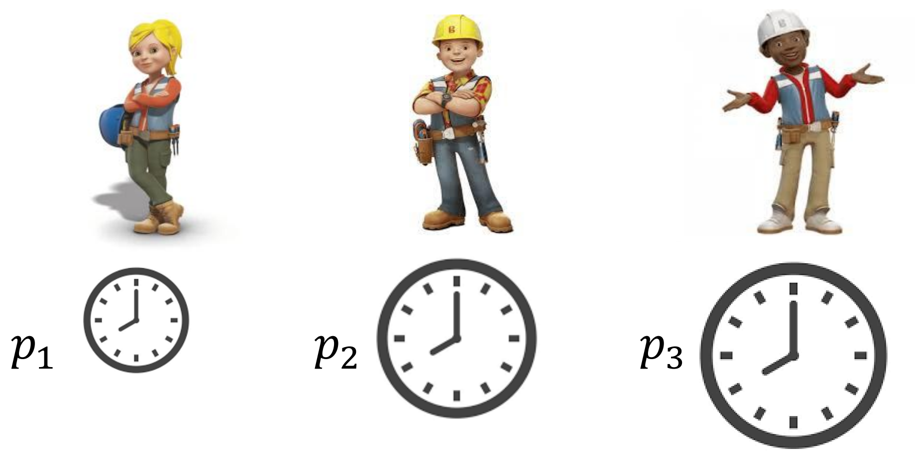
- The scheduler, being the service provider, typically decides the objective of the schedule.

- The scheduler might decide to try to be as fair as possible to all clients.

- One way to do so, is to minimize their total completion time - $\sum_j C_j$
  - which is equivalent to minimizing the average completion time of a client.

Classical Scheduling

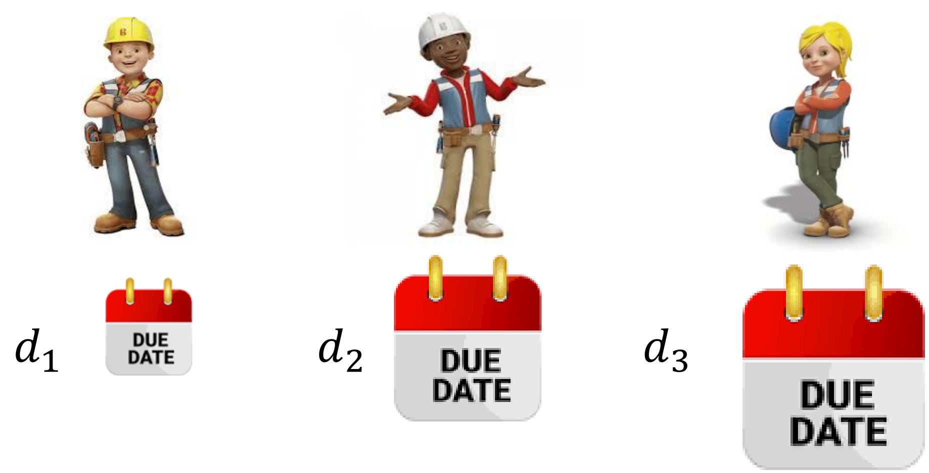
- To minimize ΣC_j, use the Shortest Processing Time first (SPT) rule:



$$\sum_j C_j = 3p_1 + 2p_2 + p_3$$

Classical Scheduling

- Suppose the clients just don’t want to be late.
- To minimize ΣU_j, use the Earliest Deadline First (EDD) rule:

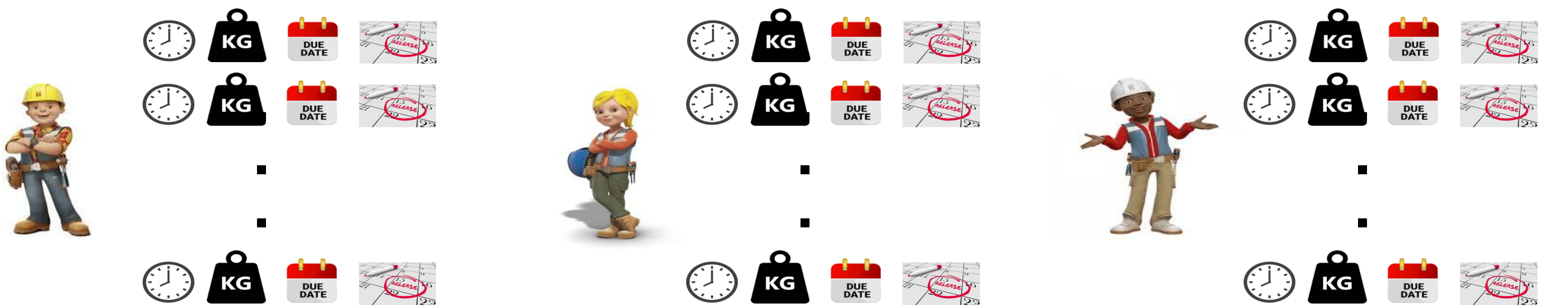


d_1 d_2 d_3

- Scan the jobs in EDD order, and when first encountering a tardy job, remove the largest job already processed (Hodson-Moore Alg.).
- Is this really fair?

Repetitive Scheduling

- But what happens when the clients keep on returning ?



- Now, in the simplest case, each client has m jobs, one per each of the m days of the month.
- Let $p_{i,j}$ denote the processing time of client j's job on day i.
- Define $C_{i,j}$ to be the completion time of client j's job on day.
- Similarly define $w_{i,j}$, $d_{i,j}$, and $r_{i,j}$ when necessary.

Repetitive Scheduling

- If we minimize the total completion time $\Sigma C_{i,j}$, a single client can complete last on every day!

- This is particularly unfair if the job processing time aren't entirely determined by the clients.
- For example:
  - Healthcare.
  - Civic duties (jury duty, military service, ect..).
  - Etc…

Repetitive Scheduling

- If we minimize the total completion time $\Sigma U_{i,j}$, a single client can be tardy every day!

- This means that from his point of view, he never gets any service!

- This is particularly unfair when the job due dates aren't entirely determined by the clients.

Fair Repetitive Scheduling

- Instead, it makes much more sense to minimize

$$\max_{j\in[n]} C_j \quad = \quad \max_{j\in[n]} \sum_{i\in[m]} C_{i,j}$$

in case the clients are interested in minimizing their completion time.

- Or, to minimize

$$\max_{j\in[n]} U_j \quad = \quad \max_{j\in[n]} \sum_{i\in[m]} U_{i,j}$$

in case the clients are interested in not being tardy.

Fair Repetitive Scheduling

- More generally, we may have any (say, minimization) objective function $F_{i,j} = f(C_{i,j},)$ that typically depends on the completion time of the clients on any given day.
- We define the (single machine) Fairness in Repetitive Scheduling problem, $1|rep|max_j\Sigma_i F_{i,j}$, as the problem of minimizing

$$\max_{j\in[n]} F_j \quad = \quad \max_{j\in[n]} \sum_{i\in[m]} F_{i,j}$$

- In the decision version, we are given a fairness threshold K, and the goal is to determine whether there exists a schedule with $\max F_j \leq K$.

Fair Repetitive Scheduling

- Thus, we obtain a performance matrix, which specifies the performance (according to the given objective function) of each job.

$F_{1,1}$	$F_{1,2}$	…	…	$F_{1,n}$
$F_{2,1}$	$F_{2,2}$	…	…	$F_{2,n}$
…	…	…		
…	…	…	…	…
…	…	…	…	…
$F_{m,1}$	$F_{m,2}$	…	…	$F_{m,n}$
F_1	F_2	…	…	F_n

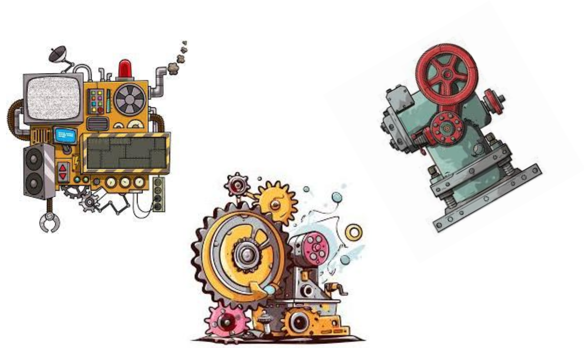
rows correspond to days

column j sums up to F_j

- The goal is to minimize the maximum value in the final row.

Fair Repetitive Scheduling

- Our model is robust, and can handle several tweaks and changes, such as

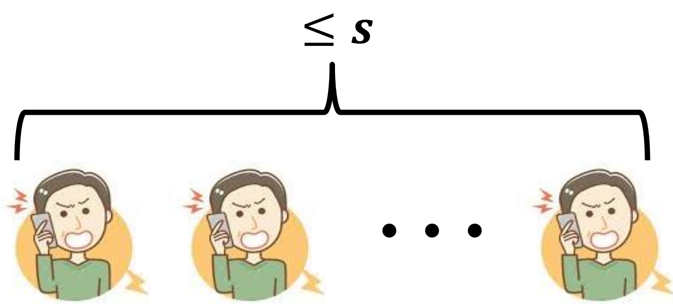


- More complex machine models.
  - E.g., the $Pq|rep|max_j\Sigma_i C_{i,j}$ problem is the problem of minimizing the maximum total completion time of every client when there are q parallel machines every day.

- Different fairness thresholds K_j for each client.
  - E.g., in cases where there are premium customers, etc.

Fair Repetitive Scheduling

- Our model allows several interesting variants to study -



- Is there a schedule which is k-fair for all but s clients?
  - A variant of approximate fairness where a solution is fair to almost everyone.

- The price of fairness?
  - The ratio between the global optimum and the worst k-fair solution.

The $1|rep|\max \Sigma C_{i,j}$ Problem

- Consider first parameter n = number of clients.

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem is (weakly) NP-hard for two or more clients* ($n \geq 2$).

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem is pseudo polynomial-time solvable for a constant number of clients* ($n = O(1)$).

- For parameter k = fairness threshold, we can show -

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem is (strongly) NP-hard even for constant fairness thresholds* ($k \geq 37$).

- Note that this shows that the problem is APX-hard, meaning it doesn't admit a PTAS (most likely).

The 1|rep|max $\Sigma C_{i,j}$ Problem

- Next consider parameter m = number of days. We prove

Theorem: *The* 1|rep|max $\Sigma C_{i,j}$ *problem is (weakly) NP-hard for four or more days* (m≥4).

Theorem: *The* 1|rep|max $\Sigma C_{i,j}$ *problem is polynomial-time solvable for two days* (m=2).

- This leads to the first open problem of the talk:

What is the complexity of 1|rep|max $\Sigma C_{i,j}$ for three days??

The Two Days Algorithm

- The algorithm is inspired by Johnson's algorithm for $F2||C_{max}$.
- We begin with a structural lemma:

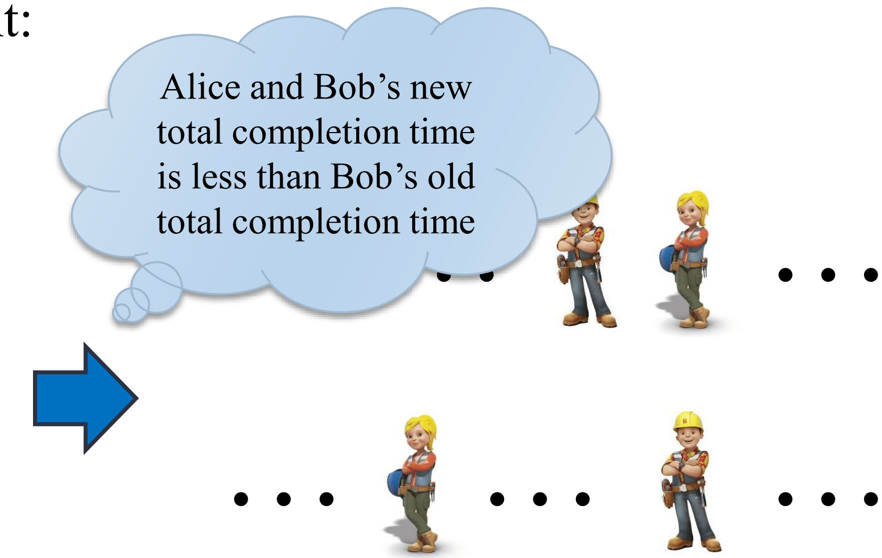
Lemma (Property 1): *There is an optimal schedule for any* $1|rep|\max \Sigma C_{i,j}$ *instance on two days, where the schedule on the second day is in reverse order of the schedule on the first day.*

- Thus, it is enough to only determine the order of processing on the first day.

The Two Days Algorithm

Lemma (Property 1): *There is an optimal schedule for any* $1|rep|\max \Sigma C_{i,j}$ *instance on two days, where the schedule on the second day is in reverse order of the schedule on the first day.*

- The proof is by an exchange argument:



The Two Days Algorithm

- Using the lemma, we construct an optimal schedule as follows:
- Partition the clients into two types:

type 1

smaller processing time on day 1 at least as large as on day 2

$p_{1,j} \leq p_{2,j}$

smaller processing time on day 2

$p_{1,j} > p_{2,j}$

type 2

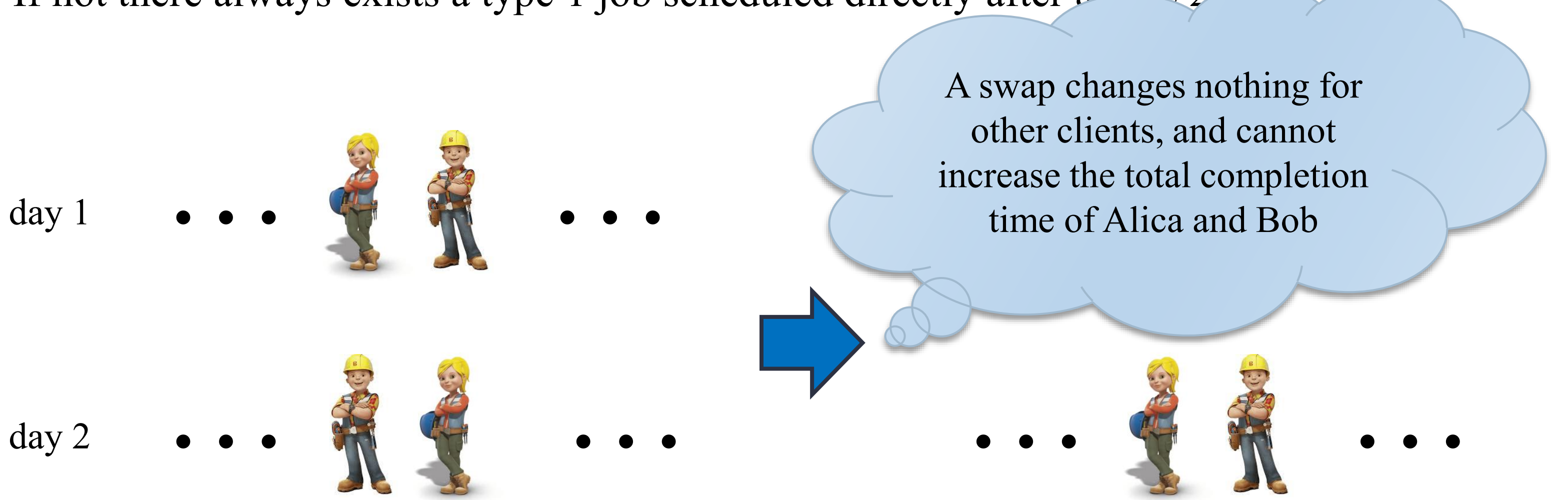
- On day 1: Schedule first the jobs of type 1 clients in SPT order, and then the jobs of type 2 clients in LPT order.
- On day 2 do the reverse of day 1.

The Two Days Algorithm

Lemma: *The schedule constructed is optimal.*

- **<u>Proof:</u>** Consider only optimal solutions where property 1 holds.
  - Step 1: Show that this set includes solutions where type 1 clients are scheduled before type 2 clients on day 1 (Property 2).

If not there always exists a type 1 job scheduled directly after a type 2 j



The Two Days Algorithm

Lemma: *The schedule constructed is optimal.*

- **<u>Proof:</u>** Consider only optimal solutions where property 1 and 2 hold.
  - Step 2: Show that this set only includes solutions where on day 1, type 1 clients are scheduled in SPT order, and type 2 clients scheduled in LPT order (Property 3).

If not, there always is a consecutive pair of clients which n.

A swap changes nothing for other clients, and the new total completion time of Alice and Charlie is at most Alice's old completion time

day 1 • • •  • • • • • •

day 2 • • •  • • • • • •  • • •

Parameterized Complexity

- We can also show the following -

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem is strongly* W[1]-*hard when parameterized by the number of days.*

- This implies that the problem is unlike to admit a $f(m)n^{O(1)}$ algorithm, even if the processing times are given in unary.
  - A $n^{f(m)}$ algorithm is still possible (yet unknown) in this case.

- One the other hand, we can show that -

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem can be solved in* $f(K+m)n^{O(1)}$ *time.*

- The proof uses n-fold ILPs.

Approximation Algorithms

- We say that a solution is α-approximate if it has a fairness threshold of k such that $k \leq \alpha \cdot OPT$, where OPT is the optimal fairness threshold.
- We show that -

Theorem: *The* $1|rep|\max \Sigma C_{i,j}$ *problem admits a 2-approximation algorithm that runs in polynomial-time.*

- While the algorithm runs in polynomial-time, it relies on several applications of an LP solver.

Is there a more efficient (combinatorial) constant approximation algorithm??

2-Approximation Algorithm

- The algorithm is inspired by 3-approximation algorithm for $1|r_j|\Sigma w_j C_j$ of Hall, Schulz, D. B. Shmoys, and Wein [MathofOR'97].
- Consider the following LP:

$$
\begin{aligned}
\min \quad & K \\
\text{s.t.} \quad & \sum_{i \in [m]} x_{i,j} \leq K && \forall\, j \in [n] \\
& \sum_{j \in S} p_{i,j} x_{i,j} \geq \frac{1}{2} \cdot P_i^2(S) && \forall\, i \in [m],\ S \subseteq [n]
\end{aligned}
$$

- The variables are:
  - $x_{i,j}$ = completion time of client j's job on day i.
  - K = the fairness threshold.

2-Approximation Algorithm

- The first set of constraints ensures that no client has total completion time which exceeds the fairness threshold k:

$$\sum_{i \in [m]} x_{i,j} \leq K \qquad \forall\, j \in [n]$$

- The second set is less clear -

$$\sum_{j \in S} p_{i,j} x_{i,j} \geq \frac{1}{2} \cdot P_i^2(S) \qquad \forall\, i \in [m],\ S \subseteq [n]$$

  - Here $P_i^2(S)$ denotes the total processing time of the jobs of S on day i, squared.

2-Approximation Algorithm

- The constraint is clearly satisfied for every singleton S={j} since

$$p_{i,j} \cdot C_{i,j} \geq p_{i,j} \cdot p_{i,j} > \frac{1}{2} \cdot p_{i,j}^2 = \frac{1}{2} \cdot P_i^2(S).$$

- And for |S|>1 we have

$$\begin{aligned} \sum_{j \in S} p_{i,j} C_{i,j} \quad &\geq \quad \sum_{j,k \in S, j \leq k} p_{i,j} p_{i,k} \quad = \quad \frac{1}{2} \cdot \Big(\sum_{j \in S} p_{i,j} \Big)^2 + \frac{1}{2} \sum_{j \in S} p_{i,j}^2 \\ &> \quad \frac{1}{2} \cdot \Big(\sum_{j \in S} p_{i,j} \Big)^2 \quad = \quad \frac{1}{2} \cdot P_i^2(S). \end{aligned}$$

- As either the job of client j is scheduled before the job of client k, or vise-versa.
- Thus, every schedule satisfies the second set of constraints.

2-Approximation Algorithm

$$
\begin{aligned}
\min \quad & K \\
\text{s.t.} \quad & \sum_{i \in [m]} x_{i,j} \le K && \forall\, j \in [n] \\
& \sum_{j \in S} p_{i,j} x_{i,j} \ge \frac{1}{2} \cdot P_i^2(S) && \forall\, i \in [m],\ S \subseteq [n]
\end{aligned}
$$

- Note that our LP has an exponential number of constraints, one for each subset of clients.
- Lucky, the LP has a separation oracle (due to Queyranne [MathProg'93]) -
  - A separation oracle is a polynomial-time algorithm that receives a solution x to the LP and determines whether x is feasible or not. If x is not feasible, the oracle returns a constraint which is violated by x.
- One can use the Primal-Dual method, in conjunction with the oracle, to obtain an optimal solution for the LP.

2-Approximation Algorithm

- Let $x^*_{1,1}, \ldots, x^*_{n,m}$ denote an optimal solution for the LP.

- Using this solution, we construct a schedule for each day i by processing the jobs in non-decreasing values of $x^*_{i,j}$.

schedule π_i
for day i

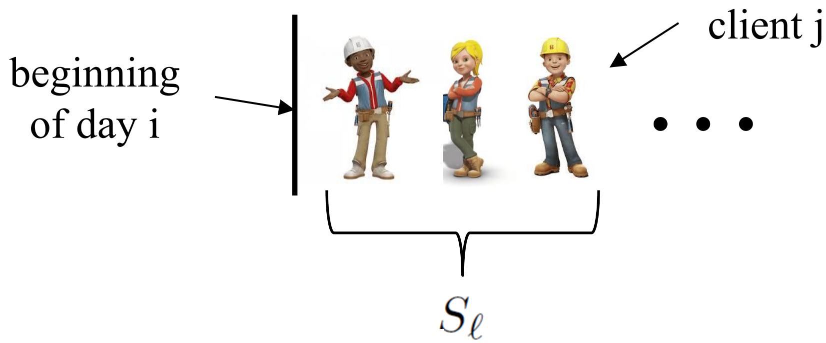
$x^*_{i,j}$ $x^*_{i,j}$ $x^*_{i,j}$

- The completion time of client j's job on day i is $C_{i,j} = \sum_{\pi_i(k) \le \pi_i(j)} p_{i,k}$.

2-Approximation Algorithm

Lemma: *The schedule constructed is 2-approximate.*

- **Proof:** Fix $i \in [m]$ and $j \in [n]$, and consider the job of client j on day i.
- Let S_ℓ denote the set of clients whose jobs were not scheduled after the job of client j (i.e., j and those before him on day i).



- Then, by our construction, we have $C_{i,j} = \sum_{\pi_i(k) \leq \pi_i(j)} p_{i,k} = P_i(S_\ell)$.

2-Approximation Algorithm

Lemma: *The schedule constructed is 2-approximate.*

- **Proof:** Fix $i \in [m]$ and $j \in [n]$, and consider the job of client j on day i.
- Then, by our construction, we have $C_{i,j} = \sum_{\pi_i(k) \leq \pi_i(j)} p_{i,k} = P_i(S_\ell)$.
- On the other hand, as $x^*_{1,1}, \ldots, x^*_{n,m}$ is feasible, and $x^*_{i,j} \geq x^*_{i,k}$ for all clients $k \in S_\ell$, we have

$$\frac{1}{2} \cdot P_i^2(S_\ell) \leq \sum_{k \in S_\ell} p_{i,k} x^*_{i,k} \leq x^*_{i,j} \cdot \sum_{k \in S_\ell} p_{i,k} = x^*_{i,j} \cdot P_i(S_\ell).$$

- It follows that $2x^*_{i,j} \geq P_i(S_\ell) = C_{i,j}$
- Thus, for any client $j \in [n]$, we get $\sum_{i=1}^{n} C_{i,j} \leq 2 \sum_{i=1}^{n} x^*_{i,j} \leq 2K^*$,

  where K^* is the optimal fairness threshold.

Further Results

- We can also show the following -

Theorem: *The* $1|\text{rep}|\max \Sigma C_{i,j}$ *problem admits a* PTAS *for a constant number of days.*

- The PTAS replies on an involved batching scheme where we batch groups of jobs together.
- We also consider the all days are the same case, and show -

Theorem: *The* $1|\text{rep},p_{i,j}=p_j|\max \Sigma C_{i,j}$ *problem admits a* $\frac{(1+\sqrt{2})}{2}$*-approximation algorithm that runs in near linear-time.*

Theorem: *The* $1|\text{rep},p_{i,j}=p_j|\max \Sigma C_{i,j}$ *problem admits a* QPTAS.

The $1|rep|\max \Sigma U_{i,j}$ Problem

- The problem is already quite hard, even when each client has the same due date on each day:

Theorem: *The* $1|rep,d_{i,j}=d_j|\max \Sigma U_{i,j}$ *problem is NP-hard, even for K=1.*

  - In the all days are the same variant, we can show for the single due date case a 1.5-approximation algorithm

- On the other hand, it is easy for unit processing times:

Theorem: *The* $1|rep,p_{i,j}=1|\max \Sigma U_{i,j}$ *problem is polynomial-time solvable.*

  - This theorem holds even when the jobs have release times, and there are multiple parallel machines for processing.

The 1|rep|max $\Sigma Z_{i,j}$ Problem

- This problem is quite hard as well:

Theorem: *The* 1|rep|max $\Sigma Z_{i,j}$ *problem is polynomial-time solvable for* $k \in \{m-1, m\}$*, and NP-hard otherwise.*

- However, when the number of clients is small -

Theorem: *The* 1|rep|max $\Sigma Z_{i,j}$ *problem can be solved in* $f(n)m^{O(1)}$ *time.*

- And when then the number of days is small -

Theorem: *The* 1|rep,di,j=dj|max $\Sigma Z_{i,j}$ *problem can be solved in* $O(mn)^{(m+1)}$ *time.*

- The all days are the same case translates directly to interval graph coloring.

Biblography

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THANK YOU FOR YOUR ATTENTION