Mixed integer linear programming for resource-constrained scheduling

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Outline

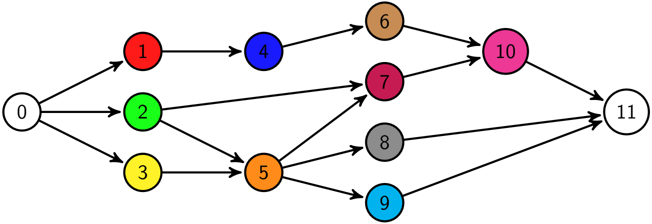
Resource-constrained project scheduling problem (RCPSP) : Introduction

- Scheduling problem with standard "finish-start" precedence constraints and resources of limited availabilities.
- Find the start time of tasks while satisfying precedence and resource constraints.
- Minimize the makespan (total project duration)

→Computationnally challenging NP-hard combinatorial optimization problem
→Generalizes many standard scheduling problem 1-machine, parallel-machines, X-shop, Assembly line balancing
→At the core of many industrial applications

The RCPSP : parameters

- R set of resources, limited constant availability $B_k \geq 0$,
- A set of activities, duration $p_i \geq 0$, resource requirement $b_{ik} \geq 0$ on each resource k,
- E set of precedence constraints (i, j), $i, j \in A$, $i < j$
- $\mathcal{T}$ time interval (scheduling horizon)



$|R| = 1, B = 4, \mathcal{T} = [0, 30)$

i	p_i	b_i
1	3	2
2	5	3
3	1	3
4	3	1
5	2	1
6	4	2
7	5	3
8	6	1
9	4	1
10	4	1

The RCPSP : variables, objective and constraints

- $S_i \geq 0$ start time of activity i
- $C_{\max}$ makespan or total project duration

RCPSP (conceptual formulation)

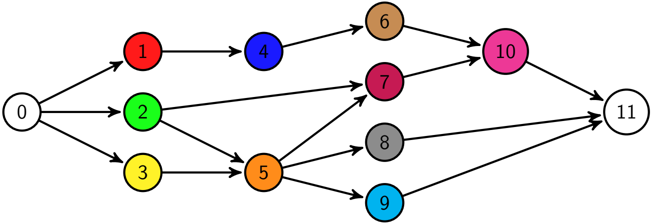
$$\min C_{\max} = \max_{i \in A} S_i + p_i$$

$$\text{s.t.} \begin{cases} S_j \geq S_i + p_i & (i,j) \in E & \textit{Precedence constraints} \\ \sum_{i \in A(t)} b_{ik} \leq B_k & t \in \mathcal{T}, k \in R & \textit{Resource constraints} \\ S_j \geq 0 & i \in A & \end{cases}$$

where $A(t) = \{j \in A | t \in [S_j, S_j + p_j)\}, \forall t \in \mathcal{T}$

The RCPSP : solution example

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



i	p_i	b_i
1	3	2
2	5	3
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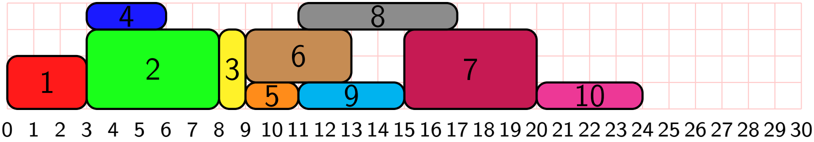
4	3	1
5	2	1
6	4	2
7	5	3
8	6	1
9	4	1
10	4	1



The Resource-Constrained Project Scheduling Problem (RCPSP)

- A central problem in many industrial applications
  - Project management, manufacturing, process industry, parallel processor architectures
- The "standard" RCPSP : An NP-hard problem posing a computational challenge since the the eighties
  - Benchmark instances [Patterson 1984], [Alvarez-Valdes and Tamarit 1989], [Kolisch, Sprecher and Drexl 1995,1997] (**PSPLIB**), [Baptiste and Le Pape 2000], [Carlier and Néron 2003] (**PACK**). [Coelho and Vanhoucke 2020]
  - 24 (out of 480) still open instances with 60 activities and 4 resources from PSPLIB

Data instances and best known results

[Vanoucke & Coelho 2018] http://solutionsupdate.ugent.be/

Table - Best known results for the RCPSP (March 2022)

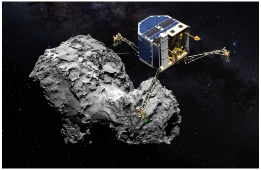
Table – Best known results for RCPSP (March 2022)

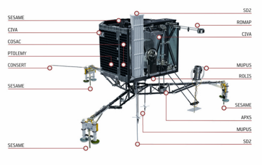
Dataset	Subset / Version	#Instances	#Open	%CPM	GAP	Observations / Results
CV	[highRD lowRU]	623	449	142.21%	3.4	CV; WZ
RG30	[highRD lowRU]	1800	116	39.27%	2.0	DH; KS; DV; CV; CV20
RG300	[highRD lowRU]	480	377	956.71%	35.2	DH; KS; DV; CV
DC1	[highRD lowRU]	1800	0	26.57%	0.0	DH; closed
DC2	[highRD lowRU]	720	210	274.20%	7.6	DH; KS; DV; CV
PSPLIB	J30 [highRD lowRU]	480	0	13.38%	0.0	DH; closed
	J60 [highRD lowRU]	480	24	10.37%	5.5	DH; KS; DV; SFSW; V; CV; C; psplib
	J90 [highRD lowRU]	480	66	9.43%	7.5	DH; KS; DV; SFSW; V; CV; psplib
	J120 [highRD lowRU]	600	290	29.01%	7.9	DH; KS; DV; SFSW; V; HKNC; CV; psplib
NetRes	NR(SP) [1k highRD lowRU]	540000 [540]	25591 [12]	78.76% [72.93%]	5.3 [1.8]	DH; KS; DV [DH; KS; DV; CV20]
	NR(AD) [1k highRD lowRU]	480000 [480]	44855 [7]	98.80% [102.43%]	5.6 [1.1]	DH; KS; DV [DH; KS; DV; CV20]
	NR(LA) [1k highRD lowRU]	720000 [720]	246 [0]	58.41% [58.87%]	4.6 [0.0]	DH; KS; DV [DH; KS; closed]
	NR(TF) [1k highRD lowRU]	720000 [720]	23563 [0]	68.28% [64.68%]	6.5 [0]	DH; KS; DV [DH; KS; DV; CV20; closed]
	NR(RC) [1k highRD lowRU]	540000 [540]	10333 [0]	66.27% [71.56%]	6.0 [0.0]	DH; KS; DV [DH; KS; closed]
	NR(RU) [1k highRD lowRU]	270000 [270]	3761 [0]	73.63% [77.00%]	9.3 [0.0]	DH; KS; DV [DH; KS; closed]
	NR(VAR) [1k highRD lowRU]	540000 [540]	4722 [0]	87.27% [91.88%]	4.3 [0.0]	DH; KS; DV [DH; KS; closed]
	VNR	1750	24	70.33%	2.3	
Patterson		110	0	18.04%	0.0	DH; closed
sD		390	229	94.10%	5.1	

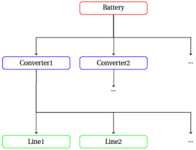
The RCPSP : complexity, variants and methods

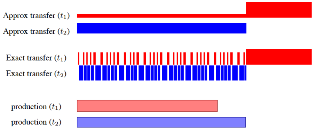
- Strongly NP-hard
- Generalizes single/parallel machine, X-shop problems
- Many relevant variants
  - Other objectives : $\min \sum_{i \in A} w_i(S_i + p_i)$
  - Generalized precedence constraints $S_j \geq S_i + l_{ij}$
  - Setup times, multiple modes, non renewable resources, preemption ...
  - Uncertainty $p_i \in [p_i^{\min}, p_i^{\max}]$, $p_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
- Exact and heuristic Methods
  - Heuristics and metaheuristics
  - Dedicated branch and bound methods
  - Specific lower bounds
  - Constraint programming (CP) or hybrid SAT/CP
  - Mixed Integer Linear Programming (MILP)
  - Large Neighborhood search (LNS)

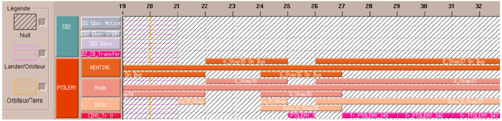
Scheduling the Philae lander experiments on the comet 67P/Churyumov–Gerasimenko











credit : CNES

- RCPSP with data transfer constraints
- 3-level Hierarchy of cumulative resource constraints
- 19 experiments, 752 activities, 926 precedence constraints,

[Simonin et al., 2012 2015] (solved by CP)

Scheduling the Airbus A330 Assembly line

credit : José Goulão, CC BY-SA 2.0

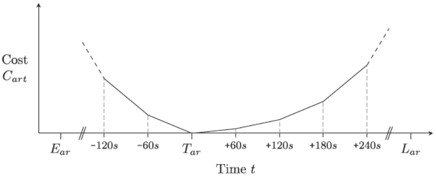
- Multi-mode RCPSP with resource leveling objective (fixed makespan of 14 to 25 days)
- About 700 activities, resources operator groups (5 to 15 operators per groupes), limited space
- [Borreguerro et al., 2021] (solved by CP-LNS)

Scheduling hazardous material examinations

credit : ASN

- Multi-skill partially preemptive RCPSP with makespan objective
- About 100 activities a week, 180 operators
- [Polo et al., 2020, 2021] (solved by CP, MILP and MILP-LNS)

Scheduling integrated runway snow removals and aircraft operation scheduling

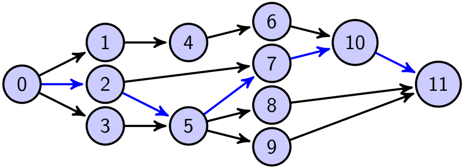


credit : John Murphy, CC BY-SA 2.0

- Parallel-machine problem with setup times
- Objective : sum of convex earliness/tardiness costs
- 3 runways, 2 snow removal groups, up to 75 aircrafts
- 2 hours planning, 40 operations per hour
- [Pohl et al., 2022] (solved by CP, MILP and hybrids)

The RCPSP : pre-processing and trivial bounds

- Upper bounds $|T|$: parallel/serial list scheduling heuristics (24)
- CPM lower bound : longest 0–$n+1$ path (16)
- Resource lower bound $\max_{k \in R} \sum_{i \in A} b_{ik} * p_i / B_k$ ($16.5 \to 17$)
- Reduce time windows $[ES_i, LS_i]$ by constraint propagation :



- Temporal constraint propagation TW
- Temporal + Resource constraint propagation TW^+

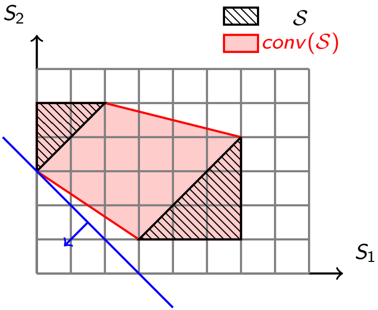
$UB = 24$ (parallel SGS / Min LFT rule)

i	p_i	b_i	TW	TW^+
1	3	2	$[0, 10]$	$[0, 10]$
2	5	3	$[0, 8]$	$[0, 6]$
3	1	3	$[0, 12]$	$[0, 12]$
4	3	1	$[3, 13]$	$[3, 13]$
5	2	1	$[5, 13]$	$[6, 13]$
6	4	2	$[6, 16]$	$[8, 16]$
7	5	3	$[7, 15]$	$[9, 15]$
8	6	1	$[7, 18]$	$[8, 18]$
9	4	1	$[7, 20]$	$[8, 20]$
10	4	1	$[12, 20]$	$[18, 20]$
11	0	0	$[16, 24]$	$[22, 24]$

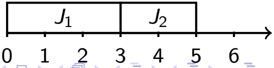
MILP for scheduling : the scheduling polyhedron

Example (release dates r_i, deadlines $\tilde{d}_i$)

$|A| = 2$, $|R| = 1$, $b_1 = b_2 = B = 1$
$p_1 = 3$, $p_2 = 2$, $r_1 = 0$, $r_2 = 1$, $\tilde{d}_1 = 9$, $\tilde{d}_2 = 7$).
Objective function $f(S) = S_1 + S_2 + p_1 + p_2$.



$$(P) \min S_1 + S_2 + 5$$
$$S_1 \geq 0$$
$$S_2 \geq 1$$
$$S_1 \leq 6$$
$$S_2 \leq 5$$
$$S_2 \geq S_1 + 3 \vee S_1 \geq S_2 + 2$$



(P) can be solved by LP on $conv(\mathcal{S})$

MILP for RCPSP : principle

- Let $\mathbf{S}$, $\mathbf{c}\,\mathbf{S}$ and $\mathcal{S}$ denote the start time vector,the linear objective and the feasible set of the RCPSP.
- Let $\mathbf{x}$ denote a vector of additional p binary variables.
- The MILP $\min_{\mathbf{S},\mathbf{x}}\{\mathbf{c}\,\mathbf{S}|\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq\mathbf{q},\mathbf{S}\geq\mathbf{0},\mathbf{x}\in\{0,1\}^p\}$ is a correct formulation for the RCPSP if we have

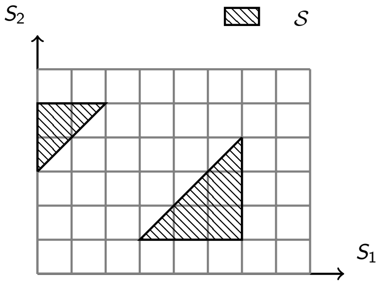
$$\mathcal{S}=\{\mathbf{S}\geq\mathbf{0}|\exists\mathbf{x}\in\{0,1\}^p,\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq\mathbf{q}\}$$

- $\mathcal{S}$ can be searched by branch and bound (and cut)
  - Branching : tree search on $\mathbf{x}$
  - Bounding : solve at each node the LP relaxation by considering unfixed $x_q\in[0,1]$ (and possibly incorporating valid inequalities)

The bound is tight if the relaxed set $\tilde{\mathcal{S}}=\{\mathbf{S}\geq\mathbf{0}|\exists\mathbf{x}\in[0,1]^p,\mathbf{M}\,\mathbf{S}+\mathbf{N}\,\mathbf{x}\leq\mathbf{q}\}$ is close to $conv(\mathcal{S})$.

MILP for RCPSP : example and issues

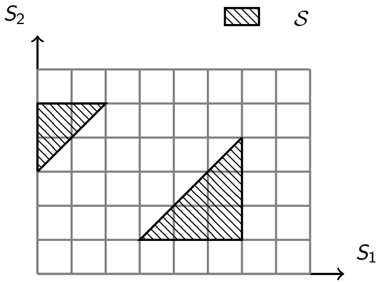
- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound



MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

$$
\begin{aligned}
(P)\ \min S_1 + S_2 + 5 \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
S_1 &\leq 6 \\
S_2 &\leq 5 \\
S_2 - S_1 + 8x &\geq 3 \\
S_1 - S_2 + 7(1-x) &\geq 2 \\
x &\in \{0,1\}
\end{aligned}
$$



The projection of the MILP feasible set on **S** maps $\mathcal{S}$

MILP for RCPSP : example and issues

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(P)\ \min S_1 + S_2 + 5 & \\
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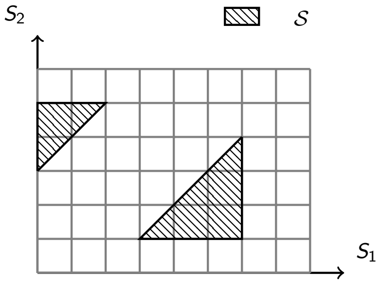

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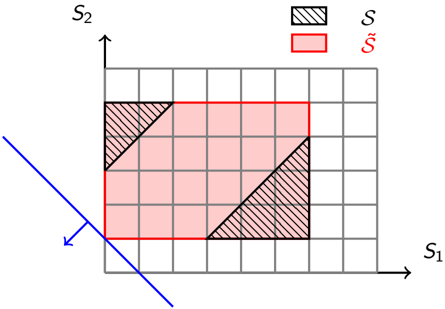


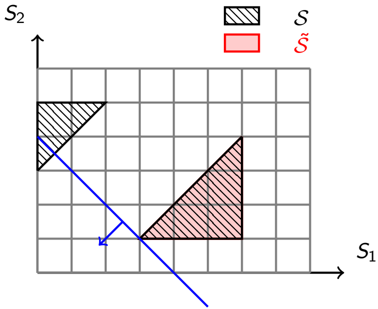
Root node LB=6

issue $x = 0.5$ always feasible

MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

$$
\begin{aligned}
(P) \min S_1 + S_2 + 5 & \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
S_1 &\leq 6 \\
S_2 &\leq 5 \\
S_2 - S_1 + 8x &\geq 3 \\
S_1 - S_2 + 7(1-x) &\geq 2 \\
x &\in \{0,1\}
\end{aligned}
$$

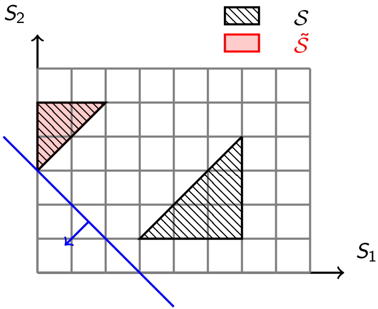


Left node $x = 1$, obj=9

MILP for RCPSP : example and issues

- Design a MIP formulation for the scheduling problem
- Solve by branch-and-bound

$$
\begin{aligned}
(P) \min S_1 + S_2 + 5 & \\
S_1 &\geq 0 \\
S_2 &\geq 1 \\
S_1 &\leq 6 \\
S_2 &\leq 5 \\
S_2 - S_1 + 8x &\geq 3 \\
S_1 - S_2 + 7(1-x) &\geq 2 \\
x &\in \{0,1\}
\end{aligned}
$$



Right node $x = 0$, obj=8

MILP for RCPSP : tradeoffs

1. Compact formulations (polynomial size)
   - Pros : fast node evaluation, mode nodes explored
   - Cons : poor LP relaxation → Branch & Cut
2. Pseudo-polynomial or extended formulations
   - Pros : obtain better LP relaxations, early node pruning in the search tree
   - Cons : increase of the MILP size (number of binary variables, constraints) towards pseudo-polynomial and even exponential sizes → Branch (& Cut) & Price techniques

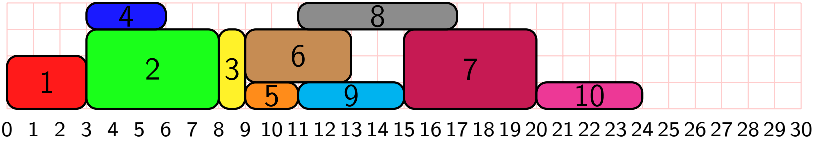
MILP for RCPSP : why ?

Scheduling problems are in general better solved by hybrid CP/SAT techniques, but :

- Tremendous progress of MILP solvers in the last years
- MILP can be preferred in identified cases (dual and primal bounds, special constraints/objectives)
- MILP can be integrated in hybrid methods (e.g. LNS)

MILP for RCPSP : families of formulations

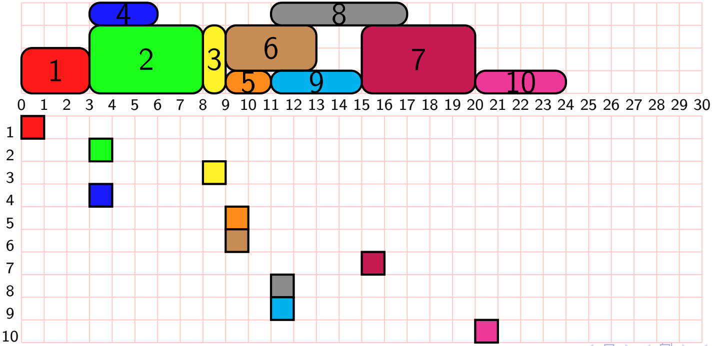
[Queyranne and Schulz 1994] classify the scheduling MILP for scheduling according to the type of decision variables, each yielding different families of valid inequalities.



1. Time-indexed variables
2. Linear-ordering variables → Strict-order or sequencing variables
3. Positional dates and assignment variables → Event-based formulations

Time-indexed pulse variables

- For integer data, $\mathcal{S}$ can be restricted to its integer vectors $\mathcal{S}^{\text{int}}$.
- "Pulse" binary variable $x_{it} = 1 \Leftrightarrow S_i = t$, for $t \in T = \mathcal{T} \cap \mathbb{N}$
- Pseudo-polynomial number of variables $|A||T|$



The aggregated time-indexed formulation

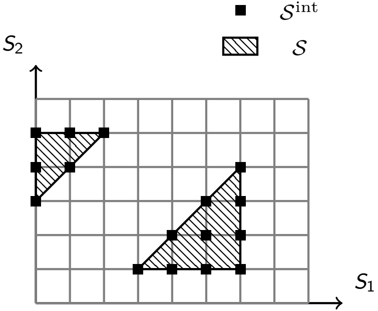
- $S_i = \sum_{t \in T} t\, x_{it}$
- $A(t) = \{i \in A | \exists \tau \in \{t - p_i + 1, \dots, t\}, x_{i\tau} = 1\}$

$$
\begin{aligned}
(DT)\ \text{Min.}\ & \sum_{t \in T} t x_{n+1,t} \\
\text{s.t.}\ & \sum_{t \in T} t x_{jt} - \sum_{t \in H} t x_{it} \geq p_i \quad (i,j) \in E \\
& \sum_{i \in V} \sum_{\tau = t - p_i + 1}^{t} b_{ik} x_{i\tau} \leq B_k \quad t \in T;\ k \in \mathcal{R} \\
& \sum_{t \in T} x_{it} = 1 \quad i \in A \\
& x_{it} \in \{0,1\} \quad i \in A
\end{aligned}
$$

[Pritsker et al. 1969]

Back to the small example : a better relaxation...

$$(P) \min S_1 + S_2 + 5$$
$$S_1 = x_{1,1} + 2x_{1,2} + 3x_{1,3} + 4x_{1,4} + 5x_{1,5} + 6x_{1,6}$$
$$S_2 = x_{2,1} + 2x_{2,2} + 3x_{2,3} + 4x_{2,4} + 5x_{2,5}$$
$$x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 1$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} = 1$$
$$x_{1,0} + x_{1,1} + x_{2,1} \leq 1$$
$$x_{2,1} + x_{2,2} + x_{1,0} + x_{1,1} + x_{1,2} \leq 1$$
$$x_{2,2} + x_{2,3} + x_{1,1} + x_{1,2} + x_{1,3} \leq 1$$
$$x_{2,3} + x_{2,4} + x_{1,2} + x_{1,3} + x_{1,4} \leq 1$$
$$x_{2,4} + x_{2,5} + x_{1,3} + x_{1,4} + x_{1,5} \leq 1$$
$$x_{2,5} + x_{1,4} + x_{1,5} + x_{1,6} \leq 1$$
$$x_{1,t} \in \{0,1\} \quad t \in \{0,\ldots,6\}$$
$$x_{2,t} \in \{0,1\} \quad t \in \{1,\ldots,5\}$$



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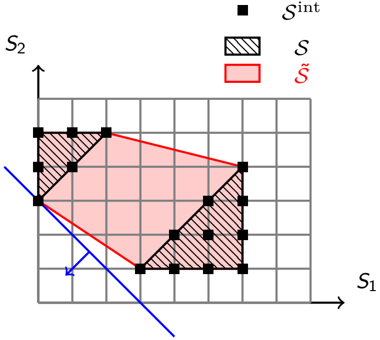
$$x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 1$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} = 1$$
$$x_{1,0} + x_{1,1} + x_{2,1} \leq 1$$
$$x_{2,1} + x_{2,2} + x_{1,0} + x_{1,1} + x_{1,2} \leq 1$$
$$x_{2,2} + x_{2,3} + x_{1,1} + x_{1,2} + x_{1,3} \leq 1$$
$$x_{2,3} + x_{2,4} + x_{1,2} + x_{1,3} + x_{1,4} \leq 1$$
$$x_{2,4} + x_{2,5} + x_{1,3} + x_{1,4} + x_{1,5} \leq 1$$
$$x_{2,5} + x_{1,4} + x_{1,5} + x_{1,6} \leq 1$$
$$x_{1,t} \in \{0,1\} \quad t \in \{0,\ldots,6\}$$
$$x_{2,t} \in \{0,1\} \quad t \in \{1,\ldots,5\}$$



In this example $\tilde{\mathcal{S}} = conv(\mathcal{S})$ and the relaxation is tight...

Back to the small example : a better relaxation...

$$(P)\ \min S_1 + S_2 + 5$$
$$S_1 = x_{1,1} + 2x_{1,2} + 3x_{1,3} + 4x_{1,4} + 5x_{1,5} + 6x_{1,6}$$
$$S_2 = x_{2,1} + 2x_{2,2} + 3x_{2,3} + 4x_{2,4} + 5x_{2,5}$$
$$x_{1,0} + x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 1$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} = 1$$
$$x_{1,0} + x_{1,1} + x_{2,1} \leq 1$$
$$x_{2,1} + x_{2,2} + x_{1,0} + x_{1,1} + x_{1,2} \leq 1$$
$$x_{2,2} + x_{2,3} + x_{1,1} + x_{1,2} + x_{1,3} \leq 1$$
$$x_{2,3} + x_{2,4} + x_{1,2} + x_{1,3} + x_{1,4} \leq 1$$
$$x_{2,4} + x_{2,5} + x_{1,3} + x_{1,4} + x_{1,5} \leq 1$$
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$$x_{1,t} \in \{0,1\} \quad t \in \{0,\ldots,6\}$$
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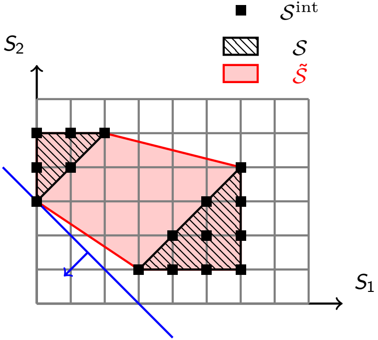
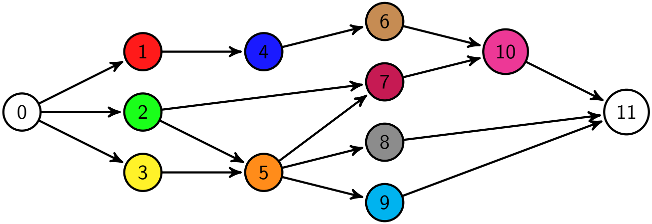


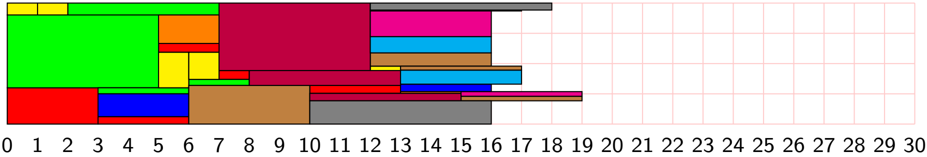
In this example $\tilde{\mathcal{S}} = conv(\mathcal{S})$ and the relaxation is tight...
... but we need 11 binary variables for a 2-task example

... but not so good in general

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



i	p_i	b_i
1	3	2
2	5	3
3	1	3
4	3	1
5	2	1
6	4	2
7	5	3
8	6	1
9	4	1
10	4	1



Bound = 16.46 (17) (not better than trivial Res. Bount)

The disaggregated time-indexed formulation (DDT)

The model can be reinforced by disaggregation of the precedence constraints, i.e. replacing precedence constraints by

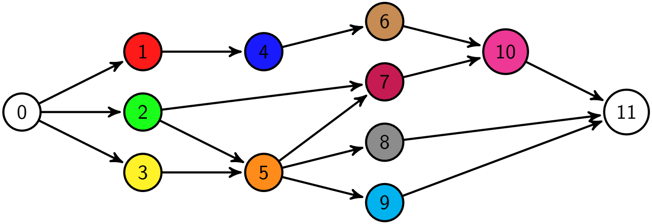
$$\sum_{\tau=0}^{t-p_i} x_{i\tau} - \sum_{\tau=0}^{t} x_{j\tau} \geq 0 \quad (i,j) \in E; \; t \in T$$

[Christofides *et al.* 1997]

- Modeling the logical relation : $S_j \leq t \Rightarrow S_i \leq t - p_i$
- The constraint matrix without resource constraints is totally unimodular.
- Total unimodularity preserved by lagrangean relaxation of the resource constraints Also efficiently computable by a max flow algorithm [Möhring *et al.* 2003]

DDT : relaxation quality

$|R| = 1, B = 4, \mathcal{T} = [0, 30)$



i	p_i	b_i
1	3	2
2	5	3
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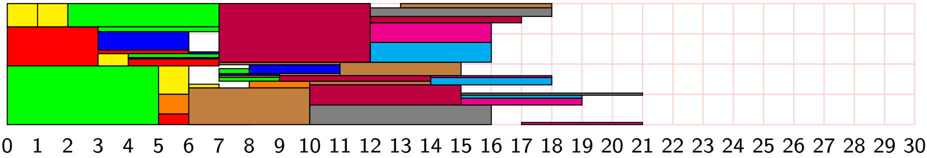
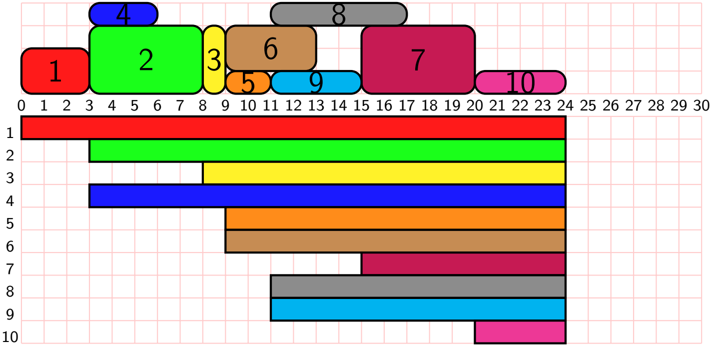
4	3	1
5	2	1
6	4	2
7	5	3
8	6	1
9	4	1
10	4	1



Bound = 17.14 (18) Strictly better than trivial bounds

Time-indexed step variables

- "Step" binary variable $\xi_{it} = 1 \Leftrightarrow S_i \leq t$, for $t \in T$
- Introduced by [Pritsker and Watters 1968] rediscovered several times... [citations removed]



Time-indexed formulations with step variables

- The time-indexed formulation with step variable (SDDT) can be obtained by (DDT) by the following transformation :

$$\xi_{it} = \sum_{\tau=0}^{t} x_{it}$$

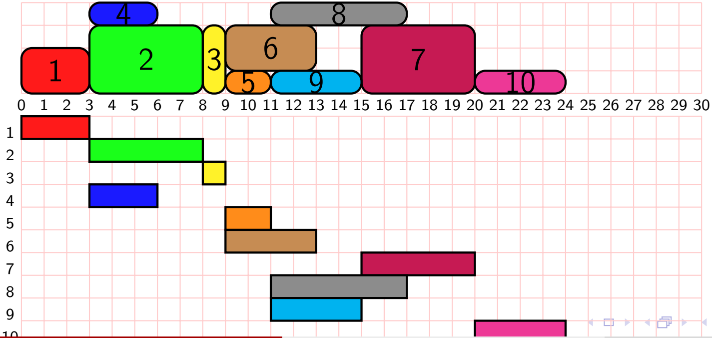
- Conversely, $x_{it} = \xi_{it} - \xi_{it-1}$
- This is a non-singular transformation (NST)
- Formulations that can be obtained from each other by a NST are strictly equivalent. They have the same $\tilde{\mathcal{S}}$ and the same relaxation value.
- [Bianco and Caramia 2013] present a variant of the step formulation based on variables $\xi'_{it} = 1 \Leftrightarrow S_i + p_i \leq t$. We can shown that it is equivalent to (SDDT) by NST [A. 2017].

On/off time-indexed step variables

- "On/off" binary variable

$$\mu_{it} = 1 \Leftrightarrow t \in [S_i, S_i + p_i[$$

- Introduced by [Lawler 1964, Kaplan 1998] for preemptive problems and [Sousa, 1989], then [Klein, 2000] and then again [Kopanos 2014] for the RCPSP.



Time-indexed formulations with on/off variables

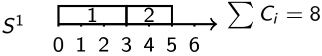
Consider the following non singular transformation [Sousa, 1989] :

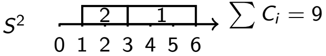
- $\mu_{it} = \sum_{\tau=t-p_i+1}^{t} x_{i\tau}$
- $x_{it} = \sum_{k=0}^{\lfloor t/p_i \rfloor} \mu_{i,t-kp_i} - \sum_{k=0}^{\lfloor (t-1)/p_i \rfloor} \mu_{i,t-kp_i-1}$
- [A. 2017] : Applying the transformation yields a time-indexed formulations with on/off variables OODDT equivalent to DDT and tighter than that of [Klein 2000] and [Kopanos 2014]
- Many "new" formulations presented in the literature are in fact weaker than or equivalent to DDT.
- Need to be distinguished from actual cutting planes or extended formulations

Extended formulations

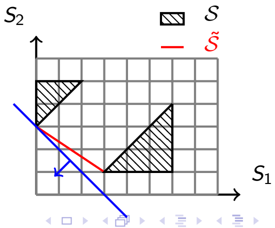
- Formulation having better relaxations...
- ... with an exponential number of constraints and/or variables
- Need to use cut and/or column generation techniques

Small example again. $\mathcal{S}^E$ dominant set of earliest schedules Let $x_s = 1$ iff schedule $S^s = \mathcal{S}^E$ is selected. $S_i = \sum_{s \in \mathcal{S}^E} S_i^s x_s$



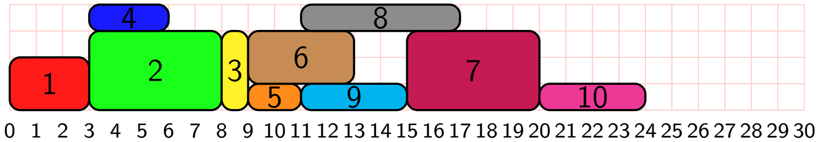


$$\min S_1 + S_2 + 5$$
$$S_1 = 3x_2$$
$$S_2 = 3x_1 + x_2$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \in \{0, 1\}$$



Forbidden sets

- Minimal forbidden set (MFS) F : a minimal set of activities that cannot be scheduled in parallel :
$\sum_{i \in F} b_{ik} > B_k$ and $\forall j \in C, \sum_{i \in F \setminus \{j\}} b_{ik} \leq B_k$



$\mathcal{F} = \{\{1,2\},\{1,3\},\{2,3\},\ldots,\{7,8,9\},\ldots\}$

- There is in general an exponential number of MFS.
- Can be reduced by excluding MFS having two activities with a precedence relation or non intersecting time windows.

Valid inequalities

- MFS-based (cover) valid inequalities [Hardin *et al* 2008]
  - Basic inequality :
    $\sum_{i\in A}\sum_{s=t-p_i+1}^{t} x_{is} \leq |F|-1, \quad \forall F \in \mathcal{F}, t \in T$
    $\rightarrow$ too many up to $O(2^n)$ $\Longrightarrow$ cut generation
  - A more general family of inequalities : extension to an interval of length v : the cover-clique inequalities

$$\sum_{i\in F\setminus\{j\}}\sum_{s=t-p_i+1+v}^{t} x_{is} + \sum_{s=t-p_j+1}^{t+v} x_{js} \leq |F|-1 \quad \forall F \in \mathcal{F}, t \in T, v \geq 0$$

Finding a minimal forbidden sets that violate such inequality (separation) is NP-hard $\Longrightarrow$ separation heuristics

- other valid inequalities [Christofides et al. 1987, de Sousa and Wolsey 1997, Cavalcante et al. 2001, Baptiste and Demassey 2004, Demassey *et al* 2005, Zhu *et al* 2006, Araujo *et al* 2020]

Lifting

$$\sum_{i \in F \setminus \{j\}} \sum_{s=t-p_i+1+v}^{t} x_{is} + \sum_{s=t-p_j+1}^{t+v} x_{js} \leq |F| - 1 \quad \forall F \in \mathcal{F}$$

The inequality defines the facets for the DT polyhedron without the precedence constraints and by setting all variables x_{ks} to 0 with $k \notin F$.

- Lifting : reinforcing the constraint by adding to the constraint variables x_{ks} with $k \notin F$
- Finding the larget α_{ks} such that

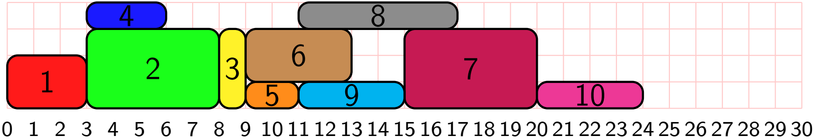
$$\sum_{i \in F \setminus \{j\}} \sum_{s=t-p_i+1+v}^{t} x_{is} + \sum_{s=t-p_j+1}^{t+v} x_{js} + \alpha_{ks} x_{ks} \leq |F| - 1 \quad \forall F \in \mathcal{F}$$

is valid.

[Hardin *et al* 2008]

Feasible subsets

- Feasible subset P : a set of activities that can be scheduled in parallel :
  $\sum_{i \in P} b_{ik} \leq B_k$ and $(i,j) \notin TA$ and
  $[ES_i, LS_i + p_i] \cap [ES_j, LS_j + p_j] \neq \emptyset$



  $\mathcal{P} = \{\{1\}, \{2\}, ..., \{10\}, \{1,5\}, \{2,4\}, \ldots, \}$

- There is in general an exponential number of FS.
- a schedule : an assignment of feasible subset to each time period
  1–2 : $\{1\}$; 3–5 : $\{2,4\}$; 6,7 : $\{2\}$; 8 : $\{3\}$; 9,10 : $\{5,6\}$; ...

The feasible subset-based formulation (FS)

[Mingozzi *et al* 1998]

- obtained from (DDT) by replacing the resource constraints by

$$\text{s. t.} \sum_{P \in \mathcal{P}_i} \sum_{t \in T} y_{Pt} = p_i \quad i \in A,\; p_i \geq 1$$

$$\sum_{P \in \overline{\mathcal{P}}} y_{Pt} \leq 1 \quad t \in T$$

$$x_i^t - \sum_{P \in \mathcal{P}_i} y_{Pt} - \sum_{P \in \mathcal{P}_i} y_{P,t-1} \geq 0 \quad i \in A;\; t \in T$$

$$y_{At} \in \{0,1\} \quad P \in \mathcal{P};\; t \in \cap_{i \in P}\{ES_i, \ldots, LS_i\}$$

where $\mathcal{P}_i \subseteq \mathcal{P}$ is the set of all feasible subsets that contain activity i.
(Dantzig-Wolfe decomposition)
Exponential number of variables → B&C&P

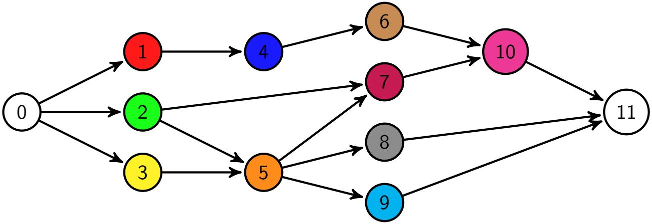
Limits of time-indexed formulations

1. Equivalent relaxations does not mean equivalent behaviour of the MILP solver for obtaining integer solutions
   - [Bianco and Caramia 2013] show that the ξ'_{it} formulation outperforms others in terms of integer solving (thanks to sparsity)
2. Even weaker relaxations may yield better integer solutions
   - Well-known that (DT) formulation may sometimes perform better than (DDT) formulation for integer solving.
3. Time-indexed formulation cannot be used for problems where large horizons are needed
   - Some examples with 15 activities are out of reach of time-indexed formulation [Kone *et al.* 2011]

Need of compact and/or hybrid formulations

Sequencing or strict ordering variable

- Principle : adding precedence constraints such that all resource conflicts are resolved
- Any schedule satisfying these new precedence constraints is feasible
- Sequencing variable $z_{ij} = 1 \Leftrightarrow S_j \geq S_i + p_i$



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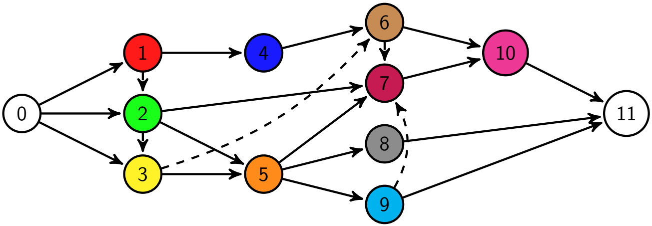

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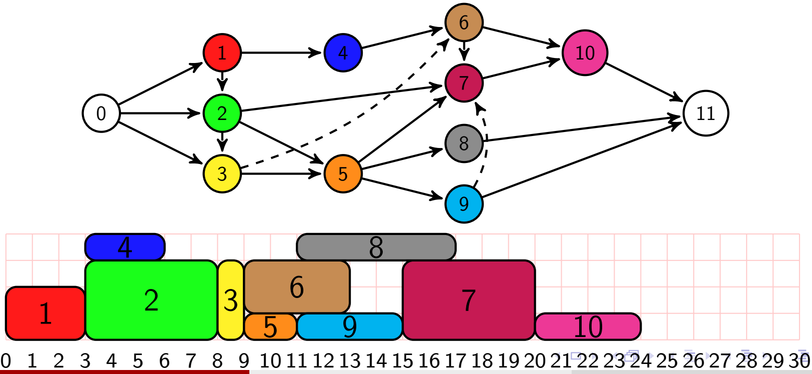


A first formulation based on forbidden sets

The set of additional precedence constraints has to “destroy” all forbidden sets.

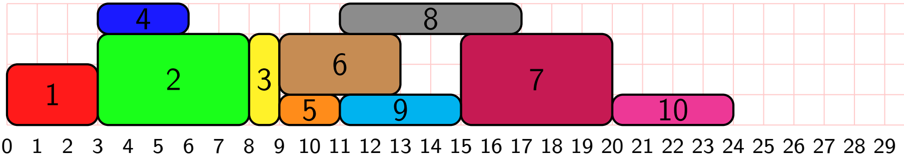
$$
\begin{aligned}
\text{Min. } & S_{n+1} \\
\text{s. t. } & z_{ij} + z_{ji} \leq 1 \quad i,j \in V,\ i < j \\
& z_{ij} + z_{jh} - z_{ih} \leq 1 \quad i,j,h \in V,\ i \neq j \neq h) \\
& z_{ij} = 1 \quad (i,j) \in E \\
& S_j - S_i + (1 - M_{ij}) z_{ij} \geq p_i \quad i,j \in V,\ i \neq j \\
& \sum_{i,j \in F, i \neq j} z_{ij} \geq 1 \quad F \in \mathcal{F} \\
& z_{ij} \in \{0,1\} \quad i,j \in V,\ i \neq j
\end{aligned}
$$

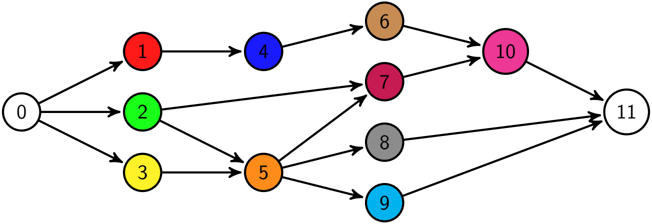
[Alvarez-Valdés and Tamarit 1993]

Extension of the disjunctive formulation for the job-shop problem [Balas 1985] with an exponential number of constraints

Resource flow variables

$\phi_{ij}^{k} \geq 0$: numbers of units of resource k transferred from i to j





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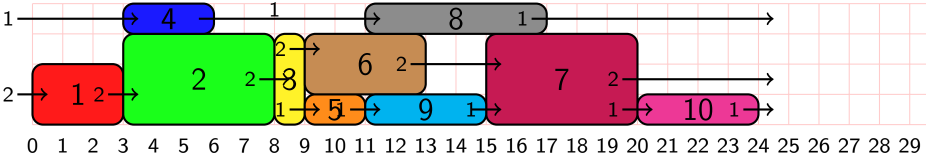

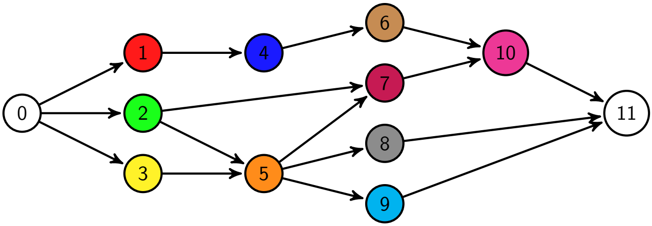

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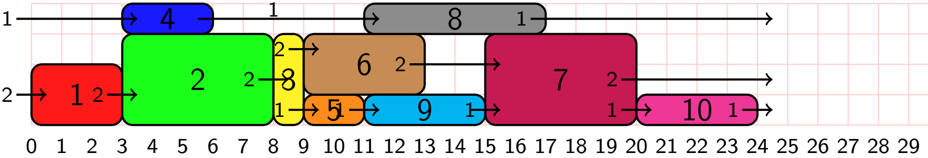
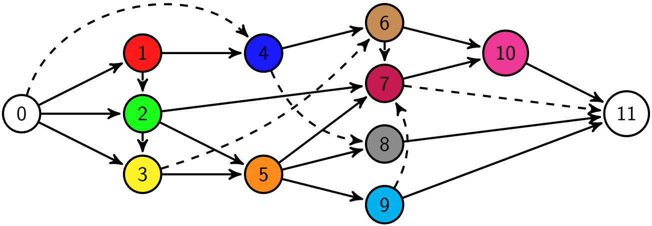


Enforcing sequencing variables to be compatible with the flow

$\phi_{ij}^k > 0 \Rightarrow z_{ij} = 1$



A formulation based on resource flows

- Replace the forbidden set constraints by the following flow constraints

$$\phi_{ij}^k - \min(\tilde{r}_{ik}, \tilde{r}_{jk}) z_{ij} \leq 0 \quad (i, j \in V,\ i \neq j,\ \forall k \in \mathcal{R})$$

$$\sum_{j \in V \setminus \{i\}} \phi_{ij}^k = \tilde{r}_{ik} \quad (i \in V \setminus \{n+1\})$$

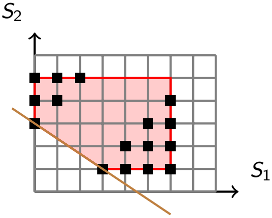
$$\sum_{i \in V \setminus \{j\}} \phi_{ij}^k = \tilde{r}_{jk} \quad (j \in V \setminus \{0\})$$

$$0 \leq \phi_{ij}^k \leq \min(\tilde{r}_{ik}, \tilde{r}_{jk}) \quad (i, j \in V,\ i \neq n+1,\ j \neq 0,\ i \neq j;\ k \in \mathcal{R})$$

- $O(|A|^2 R)$ additional continuous variables
- FB : A compact formulation. [A. *et al* 2003]

Valid inequalities for sequencing formulations

- Relaxation of poor quality, need to generate valid inequalities
- Example 1 : Extension of valid inequalities by [Balas 85,Applegate & Cook 1991,Dyer & Wolsey 1990] for the disjunctive formulation of the job-shop (half-cuts, late job cuts...)

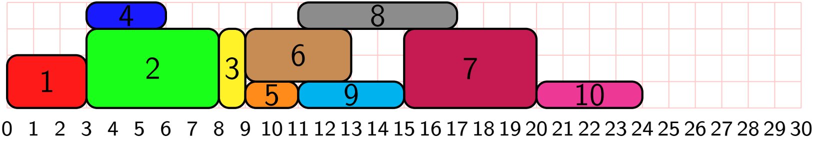


- Example 2 : constraint propagation-based cutting planes [Demassey *et al* 2005]
  - Compute conditional distances $d_{ij}^{k \prec l}$, $d_{ij}^{l \prec k}$ and $d_{ij}^{k||l}$ by CP
  - Lifted distance inequalities

$$S_j - S_i \geq d_{ij}^{h||l} + (d_{ij}^{h \prec l} - d_{ij}^{h||l}) z_{hl} + (d_{ij}^{l \prec h} - d_{ij}^{h||l}) z_{lh}$$

Start and End Event variables

- $\mathcal{E}$: set of remarkable events.
- $t_e \geq 0$: event date : representing the start and end of at least one activity
- Start binary assignment variables $a^-_{ie} = 1 \leftrightarrow S_i = t_e$
- End binary assignment variables $a^+_{ie} = 1 \leftrightarrow S_i + p_i = t_e$
- Maximum $n+1$ events $\implies 2(n+1)|\mathcal{E}|$ binary variables.



Extension of models proposed for machine scheduling [Lasserre and Queyranne 1994,Dauzère-Pérès and Lasserre 1995], widely used also in the process scheduling industry [Pinto and Grossmann 1995, Zapata *et al* 2008].

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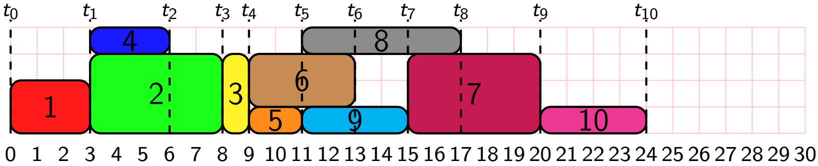


Extension of models proposed for machine scheduling [Lasserre and Queyranne 1994,Dauzère-Pérès and Lasserre 1995], widely used also in the process scheduling industry [Pinto and Grossmann 1995, Zapata *et al* 2008].

Start/End Event-based formulation (SEE)

$$\min t_n$$

$$t_0 = 0$$

$$t_f \geq t_e + p_i a_{ie}^- - p_i(1 - a_{if}^+) \qquad \forall (e,f) \in \mathcal{E}^2, f > e, \forall i \in \mathcal{J}$$

$$t_{e+1} \geq t_e \qquad \forall e \in \mathcal{E}, e < n$$

$$\sum_{e \in \mathcal{E}} a_{ie}^- = 1, \quad \sum_{e \in \mathcal{E}} a_{ie}^+ = 1 \qquad \forall i \in \mathcal{J}$$

$$\sum_{v=0}^{e} a_{iv}^+ + \sum_{v=e}^{n} a_{iv}^- \leq 1 \qquad \forall i \in \mathcal{J}, \forall e \in \mathcal{E}$$

$$\sum_{e'=e}^{n} a_{ie'}^+ + \sum_{e'=0}^{e-1} a_{je'}^- \leq 1 \qquad \forall (i,j) \in E, \forall e \in \mathcal{E}$$

$$r_{0k} = \sum_{i \in A} b_{ik} a_{i0}^- \qquad \forall k \in \mathcal{R}$$

$$r_{ek} = r_{(e-1)k} + \sum_{i \in \mathcal{J}} b_{ik} a_{ie}^- - \sum_{i \in \mathcal{J}} b_{ik} y_{ie} \qquad \forall e \in \mathcal{E}, e \geq 1, k \in R$$

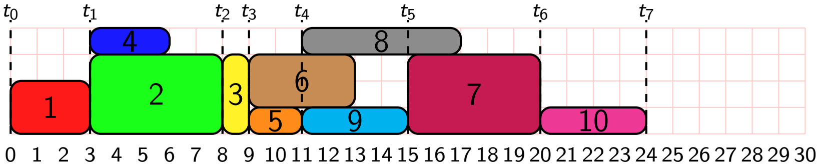
$$r_{ek} \leq B_k \qquad \forall e \in \mathcal{E}, k \in \mathcal{R}$$

$$a_{ie}^- \in \{0,1\}, a_{ie}^+ \in \{0,1\} \qquad \forall i \in \mathcal{J} \cup \{0, n+1\}, \forall e \in \mathcal{E}$$

$$t_e \geq 0, r_{ek} \geq 0 \qquad \forall e \in \mathcal{E}, k \in \mathcal{R}.$$

On/Off Event variables

- $\mathcal{E}$: set of remarkable events.
- $t_e \geq 0$: event date : representing the start of at least one activity
- On/off binary variable $aie = 1 \Leftrightarrow [S_i, S_i + p_i] \cap [t_e, t_e + 1] \neq \emptyset$
- Each activity such that $a_{ie} = 1$ can be assumed of length $[t_e, t_e + 1]$
- $n|\mathcal{E}|$ binary variables



$$
\begin{aligned}
\text{(OOE)} \quad & \text{Min. } C_{\max} \\
\text{s. t.} \quad & C_{\max} \geq t_e + (\bar{a}_{ie} - \bar{a}_{i(e-1)})p_i \quad (e \in \mathcal{E};\ i \in A) \\
& t_0 = 0 \\
& t_{e+1} \geq t_e \quad (e \neq n-1 \in \mathcal{E}) \\
& t_f \geq t_e + (\bar{a}_{ie} - \bar{a}_{i,e-1} - \bar{a}_{if} + \bar{a}_{i,f-1} - 1)p_i \quad ((e,f,i) \in \mathcal{E}^2 \times A,\ f > e \neq 0) \\
& \sum_{e'=0}^{e-1} \bar{a}_{ie'} \geq e(1 - \bar{a}_{ie} + \bar{a}_{i,e-1})) \quad (i \in A; e \neq 0 \in \mathcal{E}) \\
& \sum_{e'=e}^{n-1} \bar{a}_{ie'} \geq e(1 + \bar{a}_{ie} - \bar{a}_{i,e-1}) \quad (i \in A; e \neq 0 \in \mathcal{E}) \\
& \sum_{e \in \mathcal{E}} \bar{a}_{ie} \geq 1 \quad (i \in A) \\
& \bar{a}_{ie} + \sum_{e'=0}^{e} \bar{a}_{je'} \leq 1 + (1 - \bar{a}_{ie})e \quad (e \in \mathcal{E};\ (i,j) \in E) \\
& \sum_{i=0}^{n-1} r_{ik}\bar{a}_{ie} \leq R_k \quad (e \in \mathcal{E};\ k \in \mathcal{R}) \\
& t_e \geq 0 \quad (e \in \mathcal{E}) \\
& \bar{a}_{ie} \in \{0,1\} \quad (i \in A;\ e \in \mathcal{E})
\end{aligned}
$$

[Koné *et al.* 2011]

Valid inequalities for event-based formulations

- [Nattaf *et al.* 2019] Non-preemption inequalities for OOE

$$\sum_{q=0}^{2l}(-1)^q a_{je_q} \leq 1 \quad \forall j \in A, \forall \{e_0, \ldots, e_{2l}\} \subseteq \mathcal{E}$$

  Polynomial separation algorithm

- [Tesch 2020]
  - New valid inequalities for OOE and SEE
  - New event interval-based model IEE : variables $a_{ief} = a^-_{ie} a^+_{if}$
  - Reformulation of SEE in a LP-equivalent (but sparser) formulation $\rightarrow$ *RSEE*
  - Dominance proofs in terms of relaxation strength $OOE \prec SEE, RSEE \prec IEE$
  - Good performance of RSEE for primal and dual bounds [Koné *et al.* 2011]

MILP for solving resource-constrained scheduling problems : a few hints

- Small time horizons : use the disaggregated discrete time formulation (DDT)
- Large time horizons : use the sparse start-end event based formulation (RSEE)
- Difficulty to model some (even-linear) objective functions with event based formulations and non-linear with continuous time formulations.

Also look at instance characteristics NC (network complexity), RS (resource strength), RF (resource factor) [Kolisch *et al.* 2015] :

- large NC can narrow time windows $\Longrightarrow$ DDT
- small RS : "disjunctive resources" $\Longrightarrow$ FB better than X-E [Koné *et al.* 2011] (?)

Why using MILP for scheduling in practice ?

- Lower Bounds
  - LP relaxation of MILP formulations
  - Exact solution of preemptive or aggregated formulations
- Interest for particular cases
  - Preemption
  - Sequence-dependent setups
  - (Time-dependent) sum objective $\sum_{t \in T} w_{it} x_{it}$
- Hybrid methods
  - CP : logic-based benders decomp. [Hooker 2011], optimization-oriented global constraints [Focacci et al. 2002]...
  - LNS

Destructive lower bounds based on CP and LP

- Fix a target Makespan M. Apply CP, then LP relaxation + cuts. If M is shown infeasible, iterate with $M + 1$.
- [Demassey *et al* 2005] DT, FB + cuts
- Weighted Node packing combinatorial bound issued from the dual of the preemptive FS relaxation [Mingozzi *et al.* 1998]
- Destructive preemptive relaxation solved by constraint propagation and column generation or lagrangian relaxation [Brucker and Knust 2000, Demassey *et al* 2004, Baptiste and Demassey 2004]
- Best method [Baptiste and Demassey 2004] : use energetic reasoning cuts.

... Until Lazy Clause Generation (CP-SAT hybrid)[Schutt *et al.* 2009,2013]

MILP LB : Solving the preemptive FS exactly

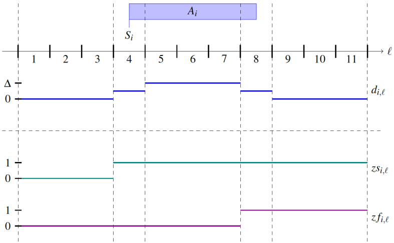
instance	LCG12	%RDDT	%DDT(1h)	PFS(3h)	instance	LCG12	%RDDT	%DDT(1h)	PFS13(3h)
j609_1	85	17.65%	2.35%		j6029_1	98	19.39%	3.06%	
j609_3	99	17.17%	9.09%		j6029_2	123	17.89%	7.32%	-3.25%
j609_5	81	14.81%	3.70%		j6029_3	114	19.30%	1.75%	-3.51%
j609_6	105	11.43%	4.76%		j6029_4	126	15.87%	7.14%	-3.17%
j609_7	105	18.10%	2.86%		j6029_5	102	12.75%	3.92%	-2.94%
j609_8	95	18.95%	7.37%		j6029_6	144	17.36%	9.03%	-1.39%
j609_9	99	12.12%	7.07%		j6029_7	117	19.66%	4.27%	
j609_10	90	15.56%	3.33%		j6029_8	98	13.27%	2.04%	-9.18%
j6013_1	105	16.19%	1.90%	-1.90%	j6029_9	105	18.10%	4.76%	
j6013_2	103	20.39%	1.94%		j6029_10	111	20.72%	1.80%	
j6013_3	84	19.05%	1.19%		j6030_2	69	4.35%	1.45%	
j6013_4	98	20.41%	3.06%		j6041_3	90	16.67%	4.44%	
j6013_5	92	21.74%	1.09%		j6041_5	109	20.18%	7.34%	
j6013_6	91	16.48%	1.10%		j6041_10	108	12.04%	2.78%	
j6013_7	83	19.28%	3.61%		j6045_1	90	12.22%	4.44%	-1.11%
j6013_8	115	20.00%	3.48%		j6045_2	134	20.90%	11.94%	-2.99%
j6013_9	97	16.49%	2.06%		j6045_3	133	13.53%	6.02%	-3.76%
j6013_10	114	24.56%	0.88%		j6045_4	101	15.84%	4.95%	-1.98%
j6025_2	95	14.74%	5.26%		j6045_5	99	21.21%	3.03%	-2.02%
j6025_4	106	18.87%	8.49%		j6045_6	132	21.97%	21.21%	-3.79%
j6025_6	105	14.29%	4.76%		j6045_7	113	19.47%	5.31%	-3.54%
j6025_7	88	15.91%	6.82%		j6045_8	119	15.13%	5.04%	-3.36%
j6025_8	95	22.11%	5.26%		j6045_9	114	16.67%	5.26%	-4.39%
j6025_10	107	15.89%	6.54%		j6045_10	102	16.67%	3.92%	-4.90%

LCG12 : [Schutt *et al* 2013] (hybrid CP/SAT method : Lazy clause generation)

PFS13 : [Moukrim *et al* 2013] Preemptive feasible subset formulation solved by B&P

MILP-based RCPSP lower bound : Solving a resource-aggregated relaxation exactly

Periodically aggregated resource-constraints [Morin *et al* 2022] (see also [Riedler *et al* 2020])
Compute the resource requirements in each interval only in time buckets taking the overlapping of task execution and the interval.



name	LB (Schutt et al., 2013)	LB $F2s+$
Pack037	116	125
Pack046	110	118
Pack050	94	100
Pack053	97	105

Reduction of time-indexed variables : can sometimes give good bounds (highly cumulative instances)

MILP-based multi-mode RCPSP

- Extension of LCG to multi mode RCPSP obtained new benchmark on this problem [Schnell *et al.* 2017]
- Strong cutting planes + Branch-and-cut improved a lot of MMRCPSP solutions [Araujo *et al.* 2020] (754 open instances solved for the first time but with > 24h computing time)

MILP-based partially preemptive multi-skill RCPSP

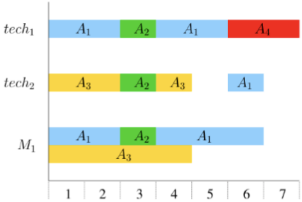


Table 4. Distribution of preemption types per set of instances.

	Set A1	Set B1	Set C1	Set D1
Non-preemptive	10%	10%	80%	33.3%
Partially preemptive	10%	80%	10%	33.3%
Preemptive	80%	10%	10%	33.3%

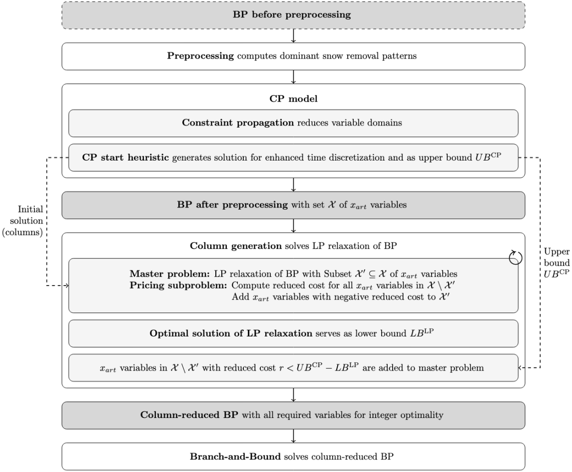
Table 6. Results of MILP and CP models after 10 min of computation using warm start

	MILP			CP		
	Number of instances solved to optimality	Average time to optimality	Average gap	Number of instances solved to optimality	Average time to optimality	Average gap
Set A1	46	87.39 s	0.05%	39	67.17 s	0.18%
Set B1	15	154.12 s	2.69%	40	88.01 s	0.15%
Set C1	0	-	9.45%	41	108.73 s	0.39%
Set D1	19	216.12 s	1.99%	40	76.14 s	0.21%
All	80	130.48 s	3.55%	160	85;27 s	0.23%

On highly preemptive instances MILP beats CP [Polo *et al.* 2020]

Hybrid MILP/CP method for runway sequencing

Aircraft landing/take-off and snow removal separation handed with clique constraints → Extended formulation



Hybrid MILP/CP method for runway sequencing : results

Table 7: Computational Times

	Time-continuous WRSP implementation (cf. Pohl et al., 2021)		Pure CP model of the WRSP			Our approach (Time-discrete WRSP using TD5e): CP start heuristic			Column generation to solve LP relaxation			BP			
Instance	Obj. Val.	*Time (s)*	Obj. Val.	CP internal Gap (%)	*Time (s)*	Obj. Val. (UB^{CP})	CP internal Gap (%)	*Time (s)*	Bound (LB^{LP})	Iterations	*Time (s)*	Obj. Val.	*Time (s)*	*Overall Time (s)*	*Improvement*
T / l / begin / 2 / 45	2,933	*6*	2,933	87	*> 300*	2,933	87	*2*	2,469	9	*1*	2,933	*4*	*7*	-
T / l / begin / 2 / 75	3,142	*40*	3,142	90	*> 300*	3,142	90	*5*	2,852	16	*4*	3,142	*11*	*20*	50%
T / l / begin / 3 / 45	1,683	*601*	1,683	100	*> 300*	1,683	100	*2*	977	9	*1*	1,683	*45*	*48*	92%
T / l / begin / 3 / 75	1,763	*1,680*	1,763	100	*> 300*	1,763	100	*6*	1,070	16	*4*	1,763	*63*	*73*	96%
T / l / cont / 2 / 45	2,933	*5*	2,933	87	*> 300*	2,933	87	*2*	2,857	12	*1*	2,933	*1*	*4*	20%
T / l / cont / 2 / 75	3,142	*26*	3,142	88	*> 300*	3,142	88	*5*	3,082	17	*4*	3,142	*3*	*12*	54%
T / l / cont / 3 / 45	664	*32*	664	100	*> 300*	664	100	*3*	533	16	*5*	664	*7*	*15*	53%
T / l / cont / 3 / 75	744	*90*	744	100	*> 300*	744	100	*6*	604	15	*7*	744	*23*	*36*	60%
E+T / l / begin / 2 / 45	1,966	*7*	2,070	92	*> 300*	2,206	92	*14*	1,637	8	*2*	2,004	*20*	*36*	-
E+T / l / begin / 2 / 75	2,133	*46*	2,508	94	*> 300*	2,492	94	*40*	1,919	13	*6*	2,170	*50*	*96*	-
E+T / l / begin / 3 / 45	1,086	*1,454*	1,125	100	*> 300*	1,167	100	*17*	748	11	*3*	1,093	*42*	*62*	96%
E+T / l / begin / 3 / 75	1,166	*2,486*	1,916	100	*> 300*	1,238	100	*48*	831	16	*7*	1,175	*101*	*156*	94%
E+T / l / cont / 2 / 45	1,966	*6*	2,136	92	*> 300*	2,349	93	*11*	1,875	8	*1*	1,999	*8*	*20*	-
E+T / l / cont / 2 / 75	2,133	*40*	2,511	94	*> 300*	2,530	94	*33*	2,078	11	*4*	2,171	*19*	*56*	-
E+T / l / cont / 3 / 45	428	*123*	435	100	*> 300*	448	100	*17*	351	13	*4*	430	*8*	*29*	76%
E+T / l / cont / 3 / 75	586	*1,503*	590	100	*> 300*	618	100	*49*	481	18	*15*	592	*45*	*109*	93%
E+T / d / begin / 2 / 45	2,373	*3*	2,917	91	*> 300*	2,686	90	*12*	1,934	8	*1*	2,413	*5*	*18*	-
E+T / d / begin / 2 / 75	2,540	*24*	3,254	93	*> 300*	3,050	93	*34*	2,206	10	*4*	2,585	*36*	*74*	-
E+T / d / begin / 3 / 45	1,247	*2,779*	1,292	100	*> 300*	1,507	100	*17*	756	10	*2*	1,261	*55*	*74*	97%
E+T / d / begin / 3 / 75	1,327	*> 3,600*	1,379	100	*> 300*	1,544	100	*49*	843	14	*6*	1,336	*104*	*159*	96%
E+T / d / cont / 2 / 45	2,373	*3*	3,113	92	*> 300*	2,839	91	*10*	2,231	9	*1*	2,429	*3*	*14*	-
E+T / d / cont / 2 / 75	2,540	*29*	3,391	93	*> 300*	3,288	93	*28*	2,461	11	*4*	2,602	*12*	*44*	-
E+T / d / cont / 3 / 45	428	*140*	436	100	*> 300*	458	100	*15*	351	15	*5*	432	*7*	*27*	81%
E+T / d / cont / 3 / 75	594	*> 3,600*	625	100	*> 300*	647	100	*46*	484	15	*12*	602	*55*	*113*	97%

All objective values rounded to integer

hybrid CP/Column generation + B&B better than CP [Pohl *et al.* 2020]

Some perspectives....

- Time aggregation / energetic reasoning / dual feasible functions [Carlier and Néron 2000, Kooli 2012]
- Mixed continuous/discrete models [Haït and A. 2012]
- Dynamic Discretization Discovery [Lagos et al 2022]
- Preprocessing [Baptiste *et al* 2010]
- B&P for the non-preemptive feasible set formulations [Foulihoux *et al* 2018]
- CG for chain decomposition models [Kimms 2001,Van den Akker *et al.* 2005
- Hybrid SAT/CP/MILP e.g. linking Clause learning and MILP [Stuckey 2010]

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